Name: Leo (Sunpeng) Li **Student Num:** 999093421

1

a) Every Gauss transform used to annihilate elements within the k-th column can be written in the following form (k = 2 given as example):

$$L_2=\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \vdots & \vdots & \vdots \\ 0 & \frac{-a_{32}}{a_{22}} & 1 & \vdots & \vdots & \vdots \\ \vdots & \frac{-a_{42}}{a_{22}} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 & 0 \\ 0 & \frac{-a_{n2}}{a_{22}} & 0 & \cdots & 0 & 1 \end{bmatrix}$$

This essentially is the square identity matrix of size n, subtracted by a matrix composed of $-a_{ik}/a_{kk}$ terms in the k-th column.

Let the column vector m_k and column vector e_k be composed like so:

$$m_k=\left[0,\cdots,\frac{a_{[k+1,k]}}{a_{[k,k]}},\frac{a_{[k+2,k]}}{a_{[k,k]}},\cdots,\frac{a_{[n,k]}}{a_{[k,k]}}\right] \qquad e_k=[0,\cdots,1,0,\cdots,0]$$

Where there are k zeros before the first term in m_k , and k-1 zeros before the 1 in e_k

Note that the result of $e_k^T m_k$ is always 0:

$e_k^T\cdot m_k=0\cdot 0+\cdots+1\cdot 0+0\cdot(a_{[k+1,k]}/a_{[k,k]})+\ldots+0\cdot(a_{[n,k]}/a_{[k,k]})=0$

And that $-m_k e_k^T=L_k-I$ gives you a square matrix of size n similar to L_k except with all the 1 terms in the diagonal subtracted, giving you 0 for the diagonal terms.

This gives us exactly what we wanted: $L_k=I-m_k e_k^T$

b) Suppose that $L_k^{-1}=I+m_k e_k^T$ and $L_k=I-m_k e_k^T$

Then it is true that:

$(I+m_k e_k^T)(I-m_k e_k^T)=I$

$\rightarrow I+(m_k e_k^T)^2=I$

$\rightarrow I+m_k e_k^T m_k e_k^T$

Note: $e_k^T m_k$ is always equal to 0 (Matrix multiplication is).
Therefore, the assumption is true, showing $L_k^{-1}=I+m_k e_k^T$ as desired.

c) Note that $(L_k L_j)^{-1}=L_j^{-1}L_k^{-1}$ (Property of matrix inverses, under the fact that both matrices are invertible)

Then from above, it is true that $L_j^{-1}L_k^{-1}=(I+m_j e_j^T)(I+m_k e_k^T)$
After simplification, $L_j^{-1}L_k^{-1}=I+m_k e_k^T+m_j e_j^T+m_j(e_j^T m_k)e_k^T$

Due to the fact that the constant term in m_k starts at the k-th position, and the 1 in e_j^T is at the j-th position(which is strictly less than k), their dot product will always be 0.

So now we have $L_j^{-1}L_k^{-1}=I+m_k e_k^T+m_j e_j^T=(I+m_k e_k^T)+(I+m_j e_j^T)-I=(L_k L_j)^{-1}$, which is equivalent to the process as described in the problem. (Changing of signs on the multipliers is identical to changing the sign in $-m_k e_k^T$ from negative to positive)

Name: Leo (Sunpeng) Li **Student Num:** 999093421

2 L and U are triangular matrices, implying that their determinant can be calculated by the product of their diagonal values. IE, the following is true: $det(T)=\prod_{i=1}^{n} t_{ii}$ where T is an upper or lower triangular matrix.

$PA=LU$

The matrix P is a modified form of the identity matrix, row switched s times in order to capture the correct row switches for matrix A, while applying Gaussian elimination.

This gives $det(P)=(-1)^{s}$, since every row switch done to a matrix causes the sign of its determinant to change.

Note that s cannot be made into the same term as k mentioned in question 1, since it is false that row switches are executed for every Gauss transform matrix L_k .

Also note that $det(P^{-1})=1/(-1)^{s}=(-1)^{s}$

Claim: $L=L_{(n-1)}^{-1}T_{(n-2)}^{-1}T_{(n-3)}^{-1}\cdots T_{1}^{-1}$ and the diagonal elements consists of all 1's.

Informal proof:

Let every Gauss transform matrix used to eliminate the k-th column defined by $L_k=I-m_k e_k^T$ have 1 in its diagonal and constants l_{ik} in the k-th column, $k<i\leq n$. Then all L_k will already have diagonal elements consisting of 1.

Let every $P_{(k+1)}$ be the n by n identity matrix, but with rows j and j+1 switched, $k<j\leq n$, since row switchs can only be performed on rows where constants l_{ik} exist.

Let every $T_k=P_{(k+1)}L_k P_{(k+1)}$

First $P_{(k+1)}$ does row switch between rows j and j+1 in L_k :

Rows j and j+1 in L_k *contains elements* l_{jk} *and* $l_{(j+1,k)}$ *, which end up being switched, along with the associated parts of the identity matrix in rows j and j+1.*

Note that the 1 in the j-th row in L_k *is now in the (j+1)-th column, and the 1 in the (j+1)-th row is now in the j-th column. (Of course, after the row switch.)*

While the second $P_{(k+1)}$ does column switch between columns j and j+1 in L_k , $k<j\leq n$:

Due to the fact that k<j, the j-th and (j+1)-th column will not contain any constants in the k-th column of L_k *. Continuing from the above, the j-th and (j+1)-th column will only contain the elementary vectors, with 1's in the (j+1)-th row for column j, and j-th row for column (j+1).*

After applying the second $P_{(k+1)}$ *, we now get* $T_k=P_{(k+1)}L_k P_{(k+1)}$ *in the form* $I-m'_k e_k^T$ *, as coveted.* ■

Claim: If all the matrices stated above have 1's in its diagonals and is lower triangular, then the product of all of them will also have 1's in the diagonals.

Informal Proof: Note that $L_{(n-1)}, T_{(n-2)}, T_{(n-3)}, \cdots, T_1$ can be written in the form of $I-m_i e_i^T$, as shown previously.

From question 1 c), we can write

$$L_{(n-1)}^{-1}T_{(n-2)}^{-1}T_{(n-3)}^{-1}\cdots T_{1}^{-1}=(L_{(n-1)}T_{(n-2)}T_{(n-3)}\cdots T_1)^{-1}=(I+m_{(n-1)}e_{(n-1)}^T)+\left[\sum_{j=1}^{n-2}(I+m'_{(j)}e_{(j)}^T)\right]-(n-2)I$$

Since $e_i^T m_i$ is always 0, It is true that m_i will always have a 0 in the position where e_i^T has a 1, causing $m_i e_i^T$ to always have 0 in the diagonal for all k. Then, it is trivial that the diagonal of $L=L_{(n-1)}^{-1}T_{(n-2)}^{-1}T_{(n-3)}^{-1}\cdots T_{1}^{-1}$ will only consist of 1, as hungered for. ■

Continued next page...

Now: $PA = LU \rightarrow A = P^{-1} LU \rightarrow det(A) = det(P^{-1}) det(L) det(U) = (-1)^s (1) \prod_{i=1}^{n} u_{ii} = (-1)^s \prod_{i=1}^{n} u_{ii}$

Where s is the number of row switches used. This is rather inexpensive to calculate, given the LU factorization with partial pivoting.