Complex Analysis

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Notes

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Consider the following

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

Definition: If $\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < R$ and diverges for all $|z| > R$, we say that the radius of convergence is R which may be infinite.

Lemma: The radius of convergence of (1) is given by $R = \frac{1}{A}$ where $A = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Suppose $\{b_n\} \leq R$. Consider the collection of all subsequences of $\{b_n\}$. Call a limit (maybe infinite) of each such subsequence, a subsequential limit. Then

$$\overline{\lim_{n\to\infty}} b_n \equiv \limsup_{n\to\infty} b_n \equiv \sup B$$

Example: $b_n = (-1)^n$ (note that this sequence is divergent). The sequence $\{b_n\}$ has two convergent subsequences, $a_n = (-1)^{2n} = 1$ and $a_n = (-1)^{2n-1} = -1$, whose respective limits are 1 and -1. Thus, $\limsup_{n\to\infty} b_n = 1$ and $\liminf_{n\to\infty} b_n = -1$.

Proof. Fix z. Then $\limsup_{n\to\infty} \sqrt[n]{|a_n||z^n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}|z| = A|z|$. If $|z| > R = \frac{1}{A}$, then $\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} > 1$. Hence, there exists infinitely many terms $\sqrt[n]{|a_n z^n|} > 1$. Thus, $|a_n z^n| > 1$ for infinitely many $n \in \mathbb{N}$. So, $|a_n z^n| \not\to 0$ as $n \to \infty$. Hence, the series diverges for $|z| > R$. (Recall that $\sum_{n=0}^{\infty} a_n z^n$ converges $\Rightarrow |a_n z^n| \to 0$ and $n \to \infty$.) If $|z| < R$, then $\exists t < 1$ such that $|a_n z^n| < t^n \forall n \geq N$ for some $N \in \mathbb{N}$. Thus, $\sum_{n=0}^{\infty} |a_n z^n| < \sum_{n=0}^{\infty} < \infty$ as a geometric series. $\square$

Now suppose that the radius of convergence of (1) is R. We claim that the radius of convergence of

$$\sum_{n=0}^{\infty} n a_n z^{n-1} \tag{2}$$

is also R. We have

$$\begin{aligned} \sum_{n=0}^{\infty} n a_n z^{n-1} &= \sum_{n=1}^{\infty} n a_n z^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \end{aligned}$$

Consider

$$
\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{(n+1)|a_{n+1})|} &= \limsup_{n\to\infty} \left(\sqrt[n]{n+1}\, \sqrt[n]{|a_{n+1}|} \right) \\
&= \underbrace{\limsup_{n\to\infty} \sqrt[n]{n+1}}_{1} \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_{n+1}|}}_{A} \\
&= 1 \cdot A = A \\
&= \frac{1}{R}
\end{aligned}
$$

Where R is the radius of convergence of (1). Indeed,

$$
\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{n+1} &= \limsup_{n\to\infty} e^{\ln(n+1)^{\frac{1}{n}}} \\
&= \limsup_{n\to\infty} e^{\frac{\ln(n+1)}{n}} \\
&= e^{\limsup_{n\to\infty} \frac{\ln(n+1)}{n}} \\
&= e^0 = 1
\end{aligned}
$$

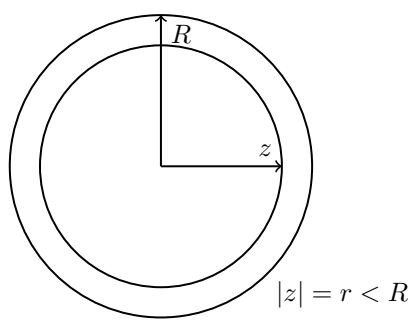
Suppose that $f(z)$ is analytic in a disk of radius R and

$$
f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3}
$$

We claim that

$$
f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} \tag{4}
$$

We know that (3) and (4) have the same radius of convergence.



Let $\varepsilon = |h|$ be such that...

Consider

$$\begin{aligned}
\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| &= \left|\frac{1}{h}\left(\sum_{k=0}^{n}\binom{n}{k}h^k z^{n-k} - z^n\right) - nz^{n-1}\right| \\
&= \left|\frac{1}{h}\sum_{k=1}^{n}\binom{n}{k}h^k z^{n-k} - nz^{n-1}\right| \\
&= \left|\sum_{k=1}^{n}\binom{n}{k}h^{k-1} z^{n-k} - nz^{n-1}\right| \\
&= \left|\sum_{k=2}^{n}\binom{n}{k}h^{k-1} z^{n-k}\right| \\
&\leq \sum_{k=2}^{n}\binom{n}{k}|h|^{k-1} r^{n-k} \\
&= \sum_{k=2}^{n}\binom{n}{k}\varepsilon^{k-1} r^{n-k} \\
&= \frac{(r+\varepsilon)^n - r^n}{\varepsilon} - nr^{n-1}
\end{aligned}$$

So we somehow get to

$$\begin{aligned}
\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| &= \left|\frac{\sum_{n=0}^{\infty} a_n (z+h)^n - \sum_{n=0}^{\infty} a_n z^n}{h} - \sum_{n=0}^{\infty} n a_n z^{n-1}\right| \\
&= \left|\sum_{n=0}^{\infty} a_n \left(\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right)\right| \\
&\leq \sum_{n=0}^{\infty} |a_n| \left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \\
&\leq \sum_{n=0}^{\infty} |a_n| \left(\frac{(r+\varepsilon)^n - r^n}{\varepsilon} - nr^{n-1}\right)
\end{aligned}$$