MT2: Tutorial question solutions

3. VECTORS

1. $\mathbf{A} = 2\mathbf{i}+\mathbf{j}-6\mathbf{k}$, $\mathbf{B} = 2\mathbf{i}+3\mathbf{j}$.

   (a)
$$|\mathbf{A}| = \sqrt{2^2+1^2+(-6)^2} = \sqrt{4+1+36} = \sqrt{41} = 6.403$$
$$|\mathbf{B}| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13} = 3.606$$

   (b)
$$\mathbf{A}+\mathbf{B} = 4\mathbf{i}+4\mathbf{j}-6\mathbf{k}$$

   (c)
$$\mathbf{A}-\mathbf{B} = -2\mathbf{j}-6\mathbf{k}$$

   (d)
$$|\mathbf{A}+\mathbf{B}| = \sqrt{4^2+4^2+6^2} = \sqrt{16+16+36} = \sqrt{68} = 8.2462$$

   (so we see that $|\mathbf{A}+\mathbf{B}| \neq |\mathbf{A}|+|\mathbf{B}|$...)

   (e)
$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{1}{\sqrt{41}}(2\mathbf{i}+\mathbf{j}-6\mathbf{k})$$
$$\hat{\mathbf{B}} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{1}{\sqrt{13}}(2\mathbf{i}+3\mathbf{j})$$
$$\widehat{\mathbf{A}+\mathbf{B}} = \frac{\mathbf{A}+\mathbf{B}}{|\mathbf{A}+\mathbf{B}|} = \frac{1}{\sqrt{68}}(4\mathbf{i}+4\mathbf{j}-6\mathbf{k})$$

2. (a) $r = \sqrt{x^2+y^2} = \sqrt{1^2+2^2} = \sqrt{5}$. $\theta = \arctan(2) = 1.11$ radians.

   (b) $r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+1^2} = \sqrt{2}$. ϕ is the angle in the xy plane, and is thus $\pi/2$, as there is no x component of this vector. θ is the zenith angle, and so is $\arccos(-z/r) = \arccos(-1/\sqrt{2}) = 3\pi/4$.

   (c) $r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$. $\theta = \arctan(2/2) = \arctan(1) = \pi/4$. $z = 3$ (the z co-ordinate is the same in cylindrical polars and Cartesians).

   (d) $x = r\cos\theta = -\sqrt{2}$, $y = r\sin\theta = \sqrt{2}$, $z = z = 2$: so $\mathbf{r} = -\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}+2\mathbf{k}$.

3. $\mathbf{p} = \mathbf{i} + 3\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{r} = -\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.
(a)

$$\begin{aligned} \mathbf{p}\cdot\mathbf{q} &= 1\times 2 + 0\times 1 + 3\times -1 = -1 \\ \mathbf{p}\cdot\mathbf{r} &= 1\times -1 + 0\times 4 + 3\times 5 = 14 \\ \mathbf{q}\cdot\mathbf{r} &= 2\times -1 + 1\times 4 - 1\times 5 = -3 \end{aligned}$$

(b)

$$\begin{aligned} |\mathbf{p}| &= \sqrt{1^2 + 3^2} = \sqrt{10} \\ |\mathbf{q}| &= \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6} \\ |\mathbf{r}| &= \sqrt{(-1)^2 + 4^2 + 5^2} = \sqrt{42} \end{aligned}$$

(c)

$$\begin{aligned} \cos\theta_{pq} &= \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = -\frac{1}{\sqrt{10}\times\sqrt{6}} \Rightarrow \theta = \arccos\left(-\frac{1}{\sqrt{60}}\right) = -1.4413 \text{ radians} \\ \cos\theta_{pr} &= \frac{\mathbf{p}\cdot\mathbf{r}}{|\mathbf{p}||\mathbf{r}|} = \frac{14}{\sqrt{10}\times\sqrt{42}} \Rightarrow \theta = \arccos\left(\frac{14}{\sqrt{420}}\right) = 0.8188 \text{ radians} \\ \cos\theta_{qr} &= \frac{\mathbf{q}\cdot\mathbf{r}}{|\mathbf{q}||\mathbf{r}|} = -\frac{3}{\sqrt{6}\times\sqrt{42}} \Rightarrow \theta = \arccos\left(-\frac{3}{\sqrt{252}}\right) = -1.3807 \text{ radians} \end{aligned}$$

4. $\mathbf{p} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{q} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$
(a)

$$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \mathbf{i}(0\times 1 - 3\times -2) - \mathbf{j}(2\times 1 - 3\times 1) + \mathbf{k}(2\times -2 - 0\times 1) = 6\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

$$\mathbf{q}\times\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 0 & 3 \end{vmatrix} = \mathbf{i}(-2\times 3 - 1\times 0) - \mathbf{j}(1\times 3 - 1\times 2) + \mathbf{k}(1\times 0 - (-2)\times 2) = -6\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

so we see, as expected, that

$$\mathbf{q}\times\mathbf{p} = -\mathbf{p}\times\mathbf{q}$$

(b)

$$\mathbf{p}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ -2 & 4 & -3 \end{vmatrix} = \mathbf{i}(0\times -3 - 3\times 4) - \mathbf{j}(2\times -3 - 3\times -2) + \mathbf{k}(2\times 4 - 0\times -2)$$

$$\Rightarrow \mathbf{p}\times\mathbf{r} = -12\mathbf{i}+8\mathbf{k} \Rightarrow |\mathbf{p}\times\mathbf{r}| = \sqrt{12^2+8^2} = 14.422$$

$$\mathbf{r}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -3 \\ 1 & -2 & 1 \end{vmatrix} = \mathbf{i}(4\times1-(-3)\times-2)-\mathbf{j}(-2\times1-(-3)\times1)+\mathbf{k}(-2\times-2-4\times1)$$

$$\Rightarrow \mathbf{r}\times\mathbf{q} = -2\mathbf{i}-\mathbf{j} \Rightarrow |\mathbf{r}\times\mathbf{q}| = \sqrt{2^2+1^2} = 2.236$$

(c)

$$\mathbf{p}\cdot(\mathbf{q}\times\mathbf{r}) = \begin{pmatrix}2\\0\\3\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix} = 2\times2+0\times-1+3\times0 = 4$$

(we know from above what $\mathbf{r}\times\mathbf{q}$ is and so can write down $\mathbf{q}\times\mathbf{r}$)

$$\mathbf{r}\cdot(\mathbf{q}\times\mathbf{p}) = \begin{pmatrix}-2\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}-6\\-1\\4\end{pmatrix} = 12-4-12 = -4$$

$$\mathbf{q}\cdot(\mathbf{r}\times\mathbf{p}) = \begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}12\\0\\-8\end{pmatrix} = 12-8 = 4$$

(all using earlier results for the cross products).

(d)
To do these we could either use the result for the vector triple product given in the lectures, or do them from first principles given that we already know the cross products in brackets. Here we do the latter.

$$\mathbf{q}\times\mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} \Rightarrow \mathbf{p}\times(\mathbf{q}\times\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 2 & 1 & 0 \end{vmatrix} = \begin{pmatrix}0\times0-3\times1\\-2\times0+3\times2\\2\times1-0\times2\end{pmatrix} = \begin{pmatrix}-3\\6\\2\end{pmatrix}$$

$$\mathbf{p}\times\mathbf{q} = \begin{pmatrix}6\\1\\-4\end{pmatrix} \Rightarrow (\mathbf{p}\times\mathbf{q})\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & -4 \\ -2 & 4 & -3 \end{vmatrix} = \begin{pmatrix}1\times-3+4\times4\\-6\times-3+-4\times-2\\6\times4-1\times-2\end{pmatrix} = \begin{pmatrix}13\\26\\26\end{pmatrix}$$

$$\mathbf{r}\times\mathbf{p} = \begin{pmatrix}12\\0\\8\end{pmatrix} \Rightarrow \mathbf{q}\times(\mathbf{r}\times\mathbf{p}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 12 & 0 & -8 \end{vmatrix} = \begin{pmatrix}-2\times-8-1\times0\\-1\times-8+1\times12\\1\times0+2\times12\end{pmatrix} = \begin{pmatrix}16\\20\\24\end{pmatrix}$$

5. (a) Following the hint, let's take $\mathbf{B} = \mathbf{i}B_x$, $\mathbf{C} = \mathbf{i}C_x+\mathbf{j}C_y$, and $\mathbf{A} = \mathbf{i}A_x+\mathbf{j}A_Y+\mathbf{k}A_z$. Then

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & 0 & 0 \\ C_x & C_y & 0 \end{vmatrix} = B_xC_y\mathbf{k}$$

(as expected, only a vector in the z-direction can be perpendicular to both **B** and **C**, which lie in the xy plane) and so

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = A_z B_x C_y$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & 0 & 0 \end{vmatrix} = A_z B_x \mathbf{j} - A_y B_x \mathbf{k}$$

so that

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = A_z B_x C_y$$

(the term in **k** disappears since there is no component of **C** in the z-direction). Finally,

$$\mathbf{A}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ C_x & C_y & 0 \end{vmatrix} = (-A_z C_y)\mathbf{i} + \ldots$$

(we don't need to work out the terms in **j** and **k** because we know already we're going to take the dot product with **B**) and so

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -B_x A_z C_y$$

which means that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})$$

QED. Since we have lost no generality in choosing our axes this way, this proves the general case too.

(b) We already found

$$\mathbf{B}\times\mathbf{C} = B_x C_y \mathbf{k}$$

and so

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ 0 & 0 & B_x C_y \end{vmatrix} = A_y B_x C_y \mathbf{i} - A_x B_x C_y \mathbf{j}$$

(necessarily this is in the xy plane since $\mathbf{B}\times\mathbf{C}$ is in the z direction). Now

$$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} = (A_x C_x + A_y C_y) B_x \mathbf{i}$$

and

$$(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (A_x B_x)(C_x \mathbf{i} + C_y \mathbf{j}) = A_x B_x C_x \mathbf{i} + A_x B_x C_y \mathbf{j}$$

So

$$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = A_x C_x B_x \mathbf{i} + A_y C_y B_x \mathbf{i} - A_x B_x C_x \mathbf{i} - A_x B_x C_y \mathbf{j} = A_y C_y B_x \mathbf{i} - A_x B_x C_y \mathbf{j}$$

which is what we obtained above, QED.

(c) Use the result from part (b):

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

which means

$$\mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = (\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$$
$$\mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}$$

So the sum of the three triple products is

$$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} + (\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} + (\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}$$

Collecting terms in **A**, **B** and **C**, this is

$$(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} + (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = 0$$

— all the terms cancel (remembering that the scalar product is commutative). QED.

6. (a) Let $\mathbf{v}_1 = \overrightarrow{AB}$, $\mathbf{v}_2 = \overrightarrow{AC}$, $\mathbf{v}_3 = \overrightarrow{BC}$. Then we have

$$\mathbf{v}_1 = \mathbf{i} - \mathbf{j}$$
$$\mathbf{v}_2 = 2\mathbf{i} - \mathbf{k}$$
$$\mathbf{v}_3 = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

(observe that $\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$ as we expect: these three vectors are not independent).

The vector equation of a line is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where **a** is a point on the line and **b** is in the direction of motion along the line. For the three lines, we have respectively that **a** is the position vector of points A, A and B, and **b** is $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In all cases $0 \le \lambda \le 1$.

(b) Any three points, which define two independent vectors, lie in a plane; the plane ABC is the plane that contains the three points A, B and C. To find a normal to the plane, we need to pick two vectors in the plane — let's choose $\mathbf{v}_1$ and $\mathbf{v}_2$ — and take the cross product:

$$\mathbf{v}_1\times\mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Let this vector be **n**: then **n** is a normal vector, but not a unit normal. We know that

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$$

and so to get a unit normal divide this by the magnitude of $\mathbf{n}$, i.e. $\sqrt{6}$:

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$$

(c) The vector equation of a plane is

$$(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0$$

where $\mathbf{n}$ is any normal to the plane and $\mathbf{a}$ is a point in the plane. Thus we can take $\mathbf{a}$ to be the position vector of point A ($\mathbf{j}+2\mathbf{k}$) and $\mathbf{n}$ to be our result from the previous section.

To test this for points A, B and C we substitute in the position vectors of those three points. For point A the equation is satisfied trivially since $\mathbf{r}-\mathbf{a}=0$. For point B, $\mathbf{r}=\mathbf{i}+2\mathbf{k}$, so we have (in column vectors)

$$\left[\begin{pmatrix}1\\0\\2\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = 1\times 1+1\times -1+0\times 2 = 0$$

For point C $\mathbf{r}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$:

$$\left[\begin{pmatrix}2\\1\\1\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = 2\times 1+0\times 1+-1\times 2 = 0$$

So, as expected, all three points satisfy the equation of the plane.