HW6, Problem 2

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We will show that $\sum_{k=1}^{n} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$ for any positive integer n, using induction on n.

Base: $n = 1$

$$\sum_{k=1}^{1} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{(1+1)} - \frac{1}{(1+2)}$$

$$\frac{2}{1^2+2} = \frac{3}{2} - \frac{1}{2} - \frac{1}{3}$$

$$\frac{2}{3} = \frac{2}{3}$$

The two are equal for $n = 1$.

Induction: Suppose that $\sum_{k=1}^{n} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{(n+1)} - \frac{1}{(n+2)}$ for $n = 1, 2, \ldots, p-1$.
We need to show $\sum_{k=1}^{p} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{(p+1)} - \frac{1}{(p+2)}$.

By the definition of summation:

$$\sum_{k=1}^{p} \frac{2}{k^2+2k} = (\sum_{k=1}^{p-1} \frac{2}{k^2+2k}) + \frac{2}{p^2+2p}$$

Our inductive hypothesis states that at $n = p - 1$:

$$\sum_{k=1}^{p-1} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{p} - \frac{1}{p+1}$$

Combining equations gives us:

$$\sum_{k=1}^{p} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{p} - \frac{1}{p+1} + \frac{2}{p^2+2p}$$

We can simplify the above equation, and the equation from our inductive hypothesis at $n = p$ to get:

$$\frac{3}{2} - \frac{1}{p} - \frac{1}{p+1} + \frac{2}{p^2+2p} = \frac{p(3p+5)}{2(p+1)(p+2)}$$

$$\frac{3}{2} - \frac{1}{p+1} - \frac{1}{p+2} = \frac{p(3p+5)}{2(p+1)(p+2)}$$

So:

$$\sum_{k=1}^{p} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{p} - \frac{1}{p+1} + \frac{2}{p^2+2p} = \frac{3}{2} - \frac{1}{p+1} - \frac{1}{p+2}$$

$$\sum_{k=1}^{p} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{p+1} - \frac{1}{p+2}$$

Since $\sum_{k=1}^{p} \frac{2}{k^2+2k} = \frac{3}{2} - \frac{1}{p+1} - \frac{1}{p+2}$, the claim is true.

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