HW6, Problem 1

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October 23, 2013

We begin with some sequence $R(n)$ defined recursively by:

$$R(0) = 0$$

$$R(1) = 3$$

$$R(2) = 11$$

$$\forall n \geq 3, R(n) = 7R(n-1) - 16R(n-2) + 12R(n-3)$$

We need to prove that $R(n) = (2n+1) \cdot 2^n - 3^n$ using strong induction on n.

Proof: by induction on n.
Base:
$R(0)$ is defined to be 0. $(2(0)+1) \cdot 2^0 - 3^0 = 2^0 - 3^0 = 0$. So $R(n) = (2n+1) \cdot 2^n - 3^n$ when $n = 0$.
$R(1)$ is defined to be 3. $(2(1)+1) \cdot 2^1 - 3^1 = 3 \cdot 2 - 3 = 3$. So $R(n) = (2n+1) \cdot 2^n - 3^n$ when $n = 1$.
$R(2)$ is defined to be 11. $(2(2)+1) \cdot 2^2 - 3^2 = 5 \cdot 4 - 9 = 11$. So $R(n) = (2n+1) \cdot 2^n - 3^n$ when $n = 2$.

Induction: Suppose that $R(n) = (2n+1) \cdot 2^n - 3^n$ for $n = 0, 1, \ldots, k-1$.

$$R(k) = 7R(k-1) - 16R(k-2) + 12R(k-3)$$

By the inductive hypothesis:

$$R(k-1) = (2(k-1)+1) \cdot 2^{k-1} - 3^{k-1} = (2k-1) \cdot 2^{k-1} - 3^{k-1}$$

$$R(k-2) = (2(k-2)+1) \cdot 2^{k-2} - 3^{k-2} = (2k-3) \cdot 2^{k-2} - 3^{k-2}$$

$$R(k-3) = (2(k-3)+1) \cdot 2^{k-3} - 3^{k-3} = (2k-5) \cdot 2^{k-3} - 3^{k-3}$$

Substituting into $R(k) = 7R(k-1) - 16R(k-2) + 12R(k-3)$ gives us:

$$R(k) = 7((2k-1) \cdot 2^{k-1} - 3^{k-1}) - 16((2k-3) \cdot 2^{k-2} - 3^{k-2}) + 12((2k-5) \cdot 2^{k-3} - 3^{k-3})$$

$$= 7((k - 2^{-1}) \cdot 2^k - 3^{-1} \cdot 3^k) - 16((k \cdot 2^{-1} - 3 \cdot 2^{-2}) \cdot 2^k - 3^{-2} \cdot 3^k) + 12((k \cdot 2^{-2} - 5 \cdot 2^{-3}) \cdot 2^k - 3^{-3} \cdot 3^k)$$

$$= (7k - 7 \cdot 2^{-1} - 16k \cdot 2^{-1} + 48 \cdot 2^{-2} + 12k \cdot 2^{-2} - 60 \cdot 2^{-3}) \cdot 2^k + (-7 \cdot 3^{-1} + 16 \cdot 3^{-2} - 12 \cdot 3^{-3}) \cdot 3^k$$

$$= (2k-1) \cdot 2^k - 3^k$$

Since $R(k) = (2k-1) \cdot 2^k - 3^k$, the claim is proven.

$$\blacksquare$$