6.6.6 Test (TS): Complex Numbers

Test

Precalculus Sem 2 (S2062436)

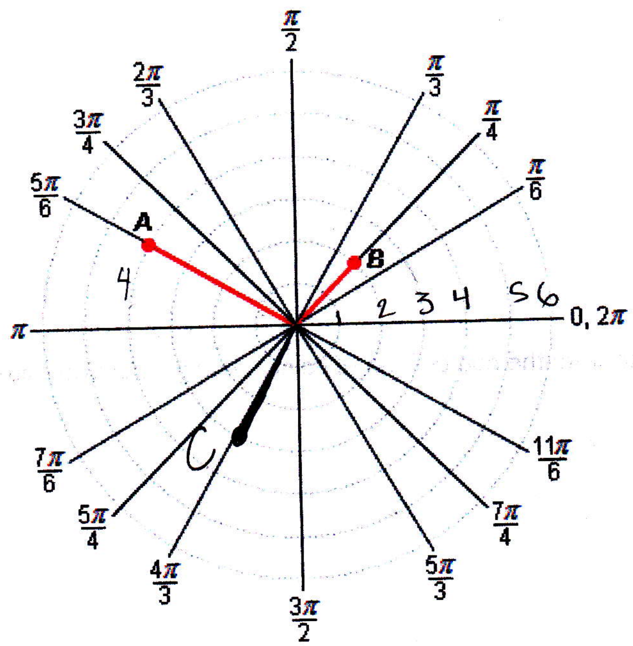
Andrew Bui

Points possible: 100

Date: ____________

Answer the following questions using what you've learned from this unit. Write your responses in the space provided.

1. Consider the polar axes with the points plotted below.



Part I: Write the coordinates of both points *A* and *B* in polar form. Be sure to write your radius as a positive value. **(4 points)**

A: $\left(4, \frac{5\pi}{6}\right)$ B: $\left(2, \pi/4\right)$

Part II: Plot the point $\left(-3, \frac{\pi}{3}\right)$ on the graph and label it as point *C*. **(2 points)**

$(r, \theta) = (-r, \theta + \pi)$

$\left(-3, \frac{\pi}{3}\right) = 3, \frac{4\pi}{3}$

Part III: Give two ways to write the coordinates of point *C* using a positive value for the radius. **(4 points)**

$(r, \theta) = (-r, \theta + \pi)$

$\left(-3, \frac{\pi}{3}\right) = \left(-(-3), \frac{\pi}{3} + \pi\right)$

$\left(-3, \frac{\pi}{3}\right) = \left(3, \frac{4\pi}{3}\right)$

2. The polar coordinates of a point are given. Find the rectangular coordinates of each point.

Part I: $\left(12, \frac{7\pi}{4}\right)$

$+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

Write your answer in the form (*x*,*y*). **(4 points)**

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$12\cos\frac{7\pi}{4}$ $\quad$ $12\sin\frac{7\pi}{4}$

$12\left(\frac{\sqrt{2}}{2}\right)$ $\quad$ $12\left(-\frac{\sqrt{2}}{2}\right)$

$= 6\sqrt{2}$ $\quad$ $= -6\sqrt{2}$

$\boxed{(6\sqrt{2}, -6\sqrt{2})}$

B. (0,7)

Use the appropriate conversion rules to write the coordinates of this point in polar (r, θ) **form. (6 points)**

$r = \sqrt{x^2+y^2}$ $\quad r = \sqrt{7^2}$

$r = 7$

$\tan\theta = \frac{7}{0}$

$\boxed{\left(7, \frac{\pi}{2}\right)}$

$\tan\theta =$ undefined

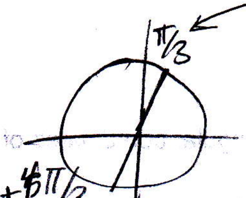
$7\sin\frac{\pi}{2} = y$

$\cos\theta = 0$, which implies $\theta = \pi/2 \pm \pi n$

$7(1) = y$

$7 = y$ ✓

line



4. Transform each polar equation to an equation in rectangular form.

A. $\theta = \frac{\pi}{3}$ $\quad r = (-\infty, \infty)$

Part I: Use the conversion rules to express *x* and *y* in terms of *r*. You should have two equations after this step. **(4 points)**

$x = r\cos\pi/3$ and $r\cos(4\pi/3)$

$y = r\sin\pi/3$ and $r\sin(4\pi/3)$

$x = \frac{1}{2}$ and $\frac{-1}{2}$ $\quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$y = \frac{\sqrt{3}}{2}$ and $\frac{-\sqrt{3}}{2}$ $\quad \left(\frac{-1}{2}, -\frac{\sqrt{3}}{2}\right)$

Part II: Use the ratio $\left(\frac{y}{x}\right)$ to find the slope of the line defined by this equation. **(2 points).**

$$\text{slope} = \frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{\frac{2\sqrt{3}}{2}}{\frac{2}{2}} \quad \text{or} \quad \frac{\frac{2\sqrt{3}}{2}}{1} \quad \text{or} \quad \frac{2\sqrt{3}}{2} \quad \text{or} \quad \sqrt{3}$$

Part III: Write the equation of the line in slope-intercept form. **(2 points)**

$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\boxed{y = x\sqrt{3}}$$

$$y = mx + b$$

$$\frac{\sqrt{3}}{2} = \frac{1}{2} \cdot \sqrt{3} + b$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} + b$$

$$0 = b$$

B. $r = -6\sin\theta$ $\quad (-6\sin\theta, \theta)$

Part 1: Rewrite this equation in terms of *x* and *y*. **(4 points)**

$$\cancel{x}\, r = 46\sin\theta$$

$$r^2 = -6r\sin\theta$$

$$r^2 = -6y$$

$$x^2 + y^2 = -6y$$

Part II: Complete the square to produce the final equation. **(4 points)**

$x^2 + y^2 = -6y$

$x^2 + y^2 + 6y = 0$

$x^2 + (y^2 + 6y + 9) = 0 + 9$

$x^2 + (y+3)^2 = 9$

Part III: Using only your answer to part II, what shape does this graph make? **(2 points)**

circle.

5. Identify, graph, and state the symmetries for the polar equation $r = 2 + 2\sin\theta$.

Part I: Based solely on the equation, what shape will this graph have? **(2 points)**

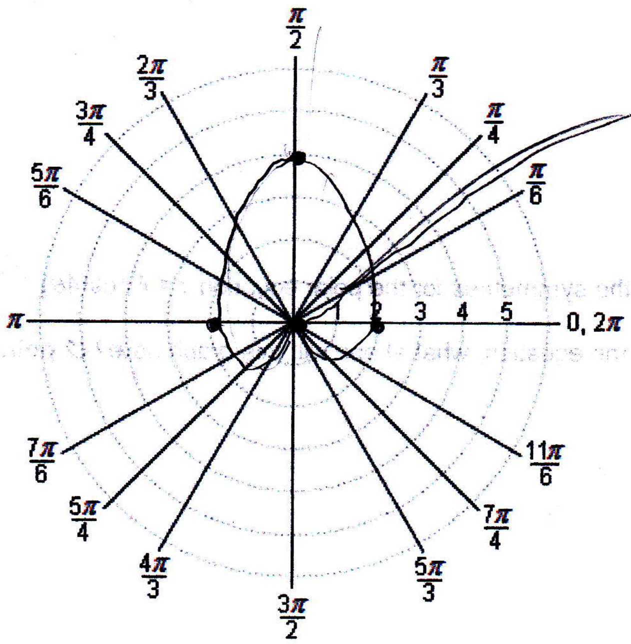
cardiod

Part II: Sketch in the graph of the function. Be sure to note and give the coordinates of at least two critical points of the graph. **(8 points)**

2+2sinθ

r	θ
2+0	0
2+2	$\frac{\pi}{2}$
2+0	π
2+-2	$\frac{3\pi}{2}$

(0, $\frac{3\pi}{2}$



(0, $\frac{3\pi}{2}$)

Critical point #1 (2, 0)

Critical point #2 (2, π)

Part III: Identify all symmetries for this graph. **(2 points)**

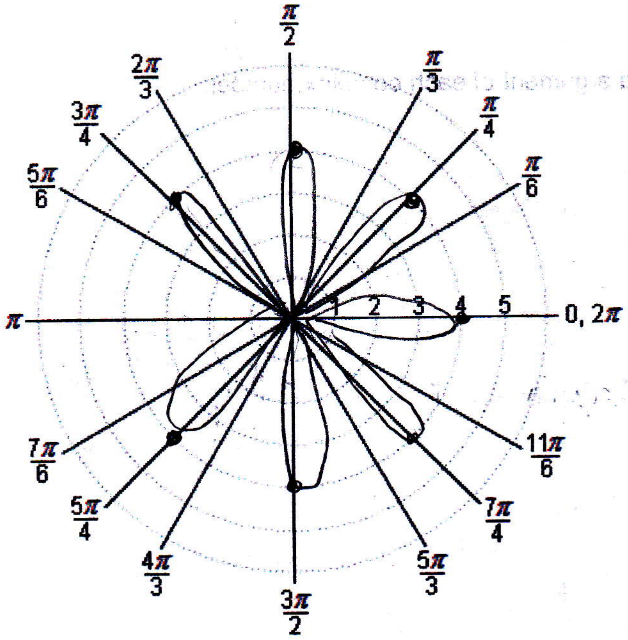
Symmetry about the lines $\theta = \frac{\pi}{2}$ and $\theta = 3\pi/2$

6. Identify, graph, and state the symmetries for the polar equation $r = 4\cos(4\theta)$.

Part I: Based solely on the equation, what shape will this graph have? **(2 points)**

Rose

Part II: Sketch in the graph of the function. Be sure to note and give the coordinates of at least two critical points of the graph. **(8 points)**



$r = 4\cos(4\theta)$

r	θ
4	0
$4\cos 2\pi$	$\frac{\pi}{2}$
$4\cos 0$	
4	
$4\cos\pi$	$\frac{\pi}{4}$
$4(-1)$	
-4	$\frac{\pi}{4}$

$(-4, 4)$

$\downarrow$

$4, \frac{5\pi}{4}$

Critical point #1 $(4, 0)$

Critical point #2 $(4, \pi/2)$

Part III: Identify all symmetries for this graph. **(2 points)**

Symmetry along the polar axis

- lines $\theta = \pi/4$ and $\theta = \frac{5\pi}{4}$
- lines $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$

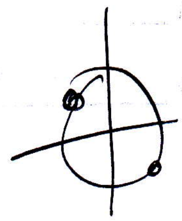
7. Compute the modulus and argument of each complex number.

Part I: $1 + i$ **(2 points)**

$\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$

modulus: $\sqrt{2}$

argument: $\arctan\left(\frac{\sqrt{2}/2}{\sqrt{2}/2}\right) = \pi/4$



Part II: $4 - 4i$ **(2 points)**

$4\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$

modulus: $4\sqrt{2}$

argument: $\arctan\left(\frac{-\sqrt{2}/2}{\sqrt{2}/2}\right) = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

Part III: $2 + 5i$ **(2 points)**

modulus $= \sqrt{(2)^2 + (5)^2}$
$= \sqrt{29}$

$\cancel{6.385\left(\cos\frac{68\pi}{180} + i\sin\frac{68.2\pi}{180}\right)}$

$\cancel{6.385\left(\cos\frac{\pi}{2.64} + i\sin\frac{\pi}{2.64}\right)}$

argument: $\arctan\frac{5}{2} = \sim 1.19$ radians

Part IV: $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ **(2 points)**

Modulus: 2

argument: arctan$\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$

$= \boxed{-1.73}$

8. Let $z = \sqrt{3} - i$ and $w = -2 - 2i$.

Part I: Convert z and w to polar form. **(4 points)**

$z = \sqrt{3} - i$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$\boxed{z = 2\left(\cos\left(\frac{7\pi}{4}\right) - i\sin\left(\frac{7\pi}{4}\right)\right)}$

$w = -2 - 2i$

$= 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$\boxed{w = 2\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) - i\sin\left(\frac{5\pi}{4}\right)\right)}$

Part II: Calculate wz using De Moivre's theorem. Express your answer in polar ($r(\cos\theta + i\sin\theta)$) form. **(4 points)**

$$wz = r_w r_z\left(\cos(\theta_r + \theta_w) + i\sin(\theta_r + \theta_w)\right)$$

$$wz = 4\sqrt{2}\left(\cos(3\pi) + i\sin(3\pi)\right)$$

$$wz = 4\sqrt{2}(-1 + i0)$$

$$wz = -4\sqrt{2}$$

Part III: Calculate z^2. Express your answer in rectangular ($a + bi$) form. **(4 points)**

$$z^2 = r^2\left(\cos(2\theta) + i\sin(2\theta)\right)$$

$$z^2 = 2^2\left(\left(\cos 2\left(\tfrac{7\pi}{4}\right) + i\sin 2\left(\tfrac{7\pi}{4}\right)\right)\right)$$

$$z^2 = 4\left(\cos\tfrac{14\pi}{4} + i\sin\tfrac{14\pi}{4}\right)$$

$$z^2 = 4\left(\cos\tfrac{7\pi}{2} + i\sin\tfrac{7\pi}{2}\right)$$

$$z^2 = 4\left(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}\right)$$

$$z^2 = 4(0 + -i)$$

$$\boxed{z^2 = -4i} \quad \text{or } (0 - 4i)$$

Part IV: Calculate w^4. Express your answer in polar ($r(\cos\theta + i\sin\theta)$) form. **(2 points)**

$w^4 = r^4(\cos 4\theta + i\sin 4\theta)$

$w^4 = 2^4\left(\cos 4\left(\frac{7\pi}{4}\right) + i\sin\left(4\left(\frac{7\pi}{4}\right)\right)\right)$

$w^4 = 16(\cos(7\pi) + i\sin(7\pi))$

$w = 16(\cos(\pi) + i\sin(\pi))$

$\boxed{w = 16(0 + i\sin\pi)}$

$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{2\pi k\theta}{n}\right)$

$+ i\sin\left(\frac{\theta + 2\pi k}{n}\right)$

9. Find the complex fourth roots of $81\left(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right)$.

Part I: Find the modulus for all of the fourth roots. **(2 points)**

81

argument

Part II: Find the angle for each of the four roots. **(6 points)**

$\arctan^{-1}\left(\frac{\sin 3\pi/8}{\cos 3\pi/8}\right) = 2.414$

Part III: Find all four of the fourth roots of $81\left(\cos\left(\frac{3\pi}{8}\right)+i\sin\left(\frac{3\pi}{8}\right)\right)$. Express your answers in polar ($r(\cos\theta + i\sin\theta)$) form. **(4 points)**

1) $\sqrt[4]{z} = \sqrt[4]{r}\left(\cos\left(\frac{2.414+2\pi 0}{4}\right) + i\sin\left(\frac{2.414+2\pi 0}{4}\right)\right)$

$= 3\left(\cos\frac{2.414}{4} + i\sin\frac{2.414}{4}\right)$

$= 3(.823 + .568i)$

$= \boxed{(2.47 + 1.703i)}$

2) $3\left(\cos\left(\frac{2.414+2\pi}{4}\right) + i\sin\left(\frac{2.414+2\pi}{4}\right)\right)$

$3(-.568 + .823i)$

$\boxed{(-1.703 + 2.47i)}$

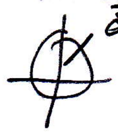
3) $3\left(\cos\left(\frac{2.414+4\pi}{4}\right) + i\sin\left(\frac{2.414+4\pi}{4}\right)\right)$ 2.414 * 2π * 2

$\boxed{-2.47 - 1.703}$

4) $3\left(\cos\left(\frac{2.414+6\pi}{4}\right) + i\sin\left(\frac{2.414+6\pi}{4}\right)\right)$ 2.414 * 2π * 3

$= \boxed{1.703 - 2.47i}$

Part II: $\left(-2,\frac{7\pi}{3}\right)$ $\frac{7\pi}{3}$ radians $=$ $\frac{\pi}{3}$ radians $\frac{1}{2}, \frac{\sqrt{3}}{2}$



Write your answer in the form (*x*,*y*). **(4 points)**

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$= -2\cos\frac{7\pi}{3}$ $\quad$ $-2\sin\frac{7\pi}{3}$

$= -2\cos\pi/3$ $\quad$ $= -2\sin\pi/3$

$= -2\left(\frac{1}{2}\right)$ $\quad$ $= -2(\sqrt{3}/2)$

$= -1$ $\quad$ $= -\sqrt{3}$

$$\boxed{(-1, -\sqrt{3})}$$

3. The rectangular coordinates of a point are given. Find the polar coordinates of each point in (r,θ) form.

A. $(-4,4\sqrt{3})$

Use the appropriate conversion rules to write the coordinates of this point in polar (r,θ) form. **(8 points)**

$r = \sqrt{x^2+y^2}$ $\quad$ $r = \sqrt{(-4)^2 + (4\sqrt{3})^2}$ $\quad$ $\tan\theta = \frac{4\sqrt{3}}{-4}$

$\tan\theta = \frac{y}{x}$ $\quad$ $r = \sqrt{16+48}$

$r = \sqrt{64}$ $\quad$ $\tan\theta = -\sqrt{3}$

$r = 8$

$$\boxed{\left(8, \frac{\pi}{3}\right) \text{ or } \left(8, \frac{4\pi}{3}\right)}$$

$\frac{\sqrt{3}}{2}$ or $\frac{1}{2}$