4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICENTS

AUXILIARY EQUATIONS

Consider the special case of the second-order DE:

$$ay'' + by' + cy = 0,$$

Where a, b, and c are constants. Substituting the solution of the form $y = e^{mx}$, we got

$$\begin{aligned} ay'' + by' + cy &= 0 \\ am^2 e^{mx} + bme^{mx} + ce^{mx} &= 0 \\ e^{mx}(am^2 + bm + c) &= 0. \end{aligned}$$

Since $e^{mx} \neq 0$, the equation $am^2 + bm + c = 0$ is called the **auxiliary equation** of the DE $ay'' + by' + cy = 0$. The roots are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ and \ m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases for the solutions of the auxiliary equation:

1. $b^2 - 4ac > 0 \implies m_1, m_2$ *are distinct real roots.*
2. $b^2 - 4ac = 0 \implies m_1, m_2$ *are repeated real roots.*
3. $b^2 - 4ac < 0 \implies m_1, m_2$ *are conjugate complex roots.*

CASE I: DISTINCT REAL ROOTS

Under the assumption that the auxiliary equation has two distinct real roots, m_1 and m_2, the linearly independent solutions are $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ on $(-\infty, \infty)$. They form the fundament set, so the general solution of the DE $ay'' + by' + cy = 0$ is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

CASE II: REPEATED REAL ROOTS

Since we get one solution, $y_1 = e^{m_1 x}$, we use the concept of reduction of order in Section 4.2 to obtain the second solution. The general solution of the DE $ay'' + by' + cy = 0$ is

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

CASE III: CONJUGATE COMPLEX ROOTS

If m_1 and m_2 are complex, then $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and β are positive real numbers. The general solution of the DE $ay'' + by' + cy = 0$ is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)xx}$$

Using the **Euler's formula:** $e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$ to work with real functions, we obtain the general solution:

$$\boxed{y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))}$$

HIGHER-ORDER EQUATIONS

To solve an n^{th}-order DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where a_i, $i = 0, 1, 2, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

Examples

1. Find the general solution of the given second-order DE:

a. $5y'' + y' = 0$ $\quad y = e^{mx}$

$e^{mx}(5m^2 + m) = 0$ $\quad e^{mx} \neq 0$,

$m(5m+1) = 0$

$m = 0, \; m = -\frac{1}{5}$

$\therefore$ The gen. sol'n is $y = c_1 e^{0x} + c_2 e^{-\frac{1}{5}x}$

b. $12y'' - 5y' - 3y = 0$

$(12m^2 - 5m - 3) = 0$ $\quad 12 \cdot 3 = 36$, $(-9)\times(4)$

$12m^2 - 9m + 4m - 3 = 0$

$3m(4m-3) + (4m-3) = 0$

$(3m+1)(4m-3) = 0$

$m = -\frac{1}{3}, \frac{3}{4}$ $\quad \therefore \; y = c_1 e^{-\frac{1}{3}x} + c_2 e^{3/4 x}$

c. $y'' + 10y' + 25y = 0$

$(m^2 + 10m + 25) = 0$

$(m+5)^2 = 0$

$m = -5$

$y = c_1 e^{-5x} + c_2 x e^{-5x}$

d. $y'' - 6y' + 10y = 0$ $\quad \sqrt{-4} = 2i$

$m = \dfrac{-(-6) \pm \sqrt{36 - 40}}{2}$

$m = 3 \pm i$, $\alpha = 3$, $\beta = 1$

The sol'n is $y = e^{3x}(c_1 \cos x + c_2 \sin x)$

e. $y''' - 3y'' - 4y' = 0$

$m^3 - 3m^2 - 4m = 0$

$m(m^2 - 3m - 4) = 0$

$m(m-4)(m+1) = 0$

$m = 0, 4, -1$

The sol'n is $y = c_1 e^{0x} + c_2 e^{4x} + c_3 e^{-x}$.

f. $y''' - 5y'' + 3y' + 9y = 0$

$m^3 - 5m^2 + 3m + 9 = 0$

$(m-1)(m^2 - 6m + 9) = 0$

$(m-1)(m-3)^2 = 0$

$m = 1, \; m = 3, 3$

The sol'n is

$y = c_1 e^{1x} + c_2 e^{3x} + c_3 x e^{3x}$

2. Solve the given initial-value problem:

a.
$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0,\ y(1) = 0,\ y'(1) = 6.$$

$m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0$

$m = 5,\ m = -1$

The gen sol$^{\underline{n}}$ is $y = C_1 e^{5x} + C_2 e^{-x} \to y' = 5C_1 e^{5x} - C_2 e^{-x}$

(*) $y(1) = 0;\quad 0 = C_1 e^5 + C_2 e^{-1}$

(**) $y'(1) = 6;\quad 6 = 5C_1 e^5 - C_2 e^{-1}$ +

$6 = 6C_1 e^5 \to C_1 = e^{-5}$

(*) $0 = (e^{-5})e^5 + C_2 e^{-1}$

$0 = 1 + C_2 e^{-1} \to C_2 = \frac{-1}{e^{-1}} = -e$

$\therefore\ y = \frac{1}{e^5} e^{5x} - e e^{-x}$

or $y = e^{5(x-1)} - e^{1-x}$

b. $y'' + 25y = 0,\ y(0) = 6,\ y'(0) = -2.$

$m^2 + 25 = 0$

$m = 0 \pm 5i$

The gen sol$^{\underline{n}}$ $y = e^{0x}(C_1 \cos 5x + C_2 \sin 5x)$

$y' = -5C_1 \sin 5x + 5C_2 \cos 5x$

(*) $y(0) = 6;\quad 6 = C_1 \cos 0 + C_2 \cancel{\sin 0} \to C_1 = 6$

(**) $y'(0) = -2;\quad -2 = -5C_1 \cancel{\sin 0} + 5C_2 \cos 0$

$-2 = 5C_2 \to C_2 = \frac{-2}{5}$

$\therefore$ The sol$^{\underline{n}}$ is $y = 6\cos 5x - \frac{2}{5}\sin 5x.$

3. Find a homogeneous linear DE with constant coefficients whose general solutions is given:

a. $y = c_1 e^{x} + c_2 e^{3x}$ $m=1,\ m=3$

$(m-1)(m-3) = 0$

$m^2 - 4m + 3 = 0$

$e^{mx}(m^2 - 4m + 3) = 0$

$\therefore\quad y'' - 4y' + 3y = 0$

b. $y = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)$, $m = \alpha \pm \beta i = -1 \pm 2i$

or $y = e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right)$

$\alpha = -1 \qquad \beta = 2$

$(m - (-1+2i))(m - (-1-2i)) = 0$

$(m + 1 - 2i)(m + 1 + 2i) = 0$

$(m+1)^2 - (2i)^2 = 0$

$m^2 + 2m + 1 - 4i^2 = 0$ $\quad -4(-1)$

$m^2 + 2m + 5 = 0$

$y'' + 2y' + 5y = 0$

Check

$\dfrac{-2 \pm \sqrt{4 - 20}}{2}$

$= \dfrac{-2}{2} \pm \dfrac{4i}{2}$

$= -1 \pm 2i$ ✓

c. $y = c_1 + c_2 x + c_3 e^{3x}$

or $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{3x}$, $m = 0, 0, 3$

$(m)(m)(m-3) = 0$

$m^3 - 3m^2 = 0$

$y''' - 3y'' = 0$

HOMEWORK PROBLEMS FOR SECTION 4.3

1-58: all odds