Basic Algebraic Skills

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The arrangements have been changed, while olympiad materials will be added after Ch.7.

- It's a set of notes for junior players of olympiad maths. They should have a basic concept of mathematics and variable.
- This set of notes is not expected to serve as a complete guide for the competitions, and practice within and without the book exercise is greatly encouraged.
- Short nontation will be most likely avoided in this set of note except $\iff$. It means if and only if, or equivalent to.
- Starting from the later part of Ch.5 and 6, pure mathematics is invloved and a F.6 pure mathematics textbook can be useful.
- If there're any questions or problems, welcome to ask through my blog http://allmaths.blogspot.com/

Objective:

- To acquire ability to understand how to analysis and the way to complete a question.
- To obtain knowledge outside the normal lesson provided.

Prerequisite:

- A basic concept of variables and ability of systematic thinking.
- A basic concept of coordinate geometry is required for some chapter: Chapter 5.3 and 6.2

Contents

Chapter 1

Polynomial

1.1 Introduction

Definition 1.1.1 *A term of variable x is in form of ax^c, where a is a non-zero constant and c is a non-negitive integer. If $c = 0$, then it reduces to constant term.*

Definition 1.1.2 *The coefficient of ax^c is a while it's degree is c.*

Definition 1.1.3 *Polynomial is the sum of terms with different terms. A polynomial with degree n and variable x is in the form of*

$$\sum_{i=0}^{n} a_i x^i = a_0x^0 + a_1x^1 + ... + a_nx^n$$

Definition 1.1.4 *Degree of a polynomial is defined as the highest degree among the terms.*

Example 1.1.5 $2x^2 + 4x + 1$ *has degree 2, and the coefficient of x^2 is 2.*

1.2 Arithmetic of polynomial

All aritemetic of polynomial obeys all laws of aritemetic of numbers. For example, $a(b + c) = ab + ac$. The laws play an important role when we deal with addition, subtraction, multiplication and division of polynomial. We have to familiar with the indice law first before we can do the arithemetic of polynomials.

Theorem 1.2.1 *The two important indice laws:*

- $x^{a+b} = (x^a)(x^b)$
- $\sqrt[b]{x^a} = x^{\frac{a}{b}}$

In fact, there're several indice laws, but the above two is enough for readers to derive the remainings. By $a(b+c) = ab + ac$, we can expand the polynomial from a from of products of polynomials.

We can multiply a constant to a polynomial.

Example 1.2.2 $2(x^7 + 3x^2 + 5) = 2x^7 + 6x^2 + 10$

We can also multiply a term to the polynomial.

Example 1.2.3 $x^3(x^5 + 3x^2 + 5) = x^8 + 3x^5 + 5x^3$

Then we can multiply polynomial by polynomial.

Example 1.2.4 $(x+2)(3x+7) = x(3x+7) + 2(3x+7) = 3x^2 + 7x + 6x + 14 = 3x^2 + 13x + 14$

Like terms means that the variable part of two terms is EXACTLY the same. The coefficient can be different.

Unlike terms means that the variable part is different.

Example 1.2.5 $4x$ *and* $6x$ *are like terms, but* $6x$ *and* $6y$ *are unlike terms.*

Polynomials can be conducted in more than one variables.

Example 1.2.6 $x(y + 3x^2) = xy + 3x^3$

Exercises

1. Given a polynomial $3x^2 + 7x - 6$, determine the degree of the polynomial, as well as the coefficient of x and the constant term.
2. Expand $x(3x^2 + 2x + 1)$.
3. Expand $(x + 2)(8x - 1)$.
4. Expand $(x^2 + 1)(x + 2)$.
5. Expand $(x + y - 2)^2$
6. Derive the indice law $x^{a-b} = \frac{x^a}{x^b}$
7. Derive the indice law $x^{-a} = \frac{1}{x^a}$
8. Derive the indice law $x^0 = 1$.
9. Derive the indice law $(x^a)^b = x^{ab}$.

1.3 Substitution

It's an exetremely important skill in mathematics.

In different formulas, we often substitute a variable by a number, or another set of variables.

Example 1.3.1 *If* $u = x+2$*, then* $(u-1)^2 = [(x+2)-1]^2 = (x+1)^2 = x^2+2x+1$*. In another way,* $(u-1)^2 = u^2-2u+1 = (x+2)^2-2(x+2)+1 = x^2+2x+1$*.*

Skill: In mathematical formula, the same representation for different formulas may not represent the same values unless they're specified.

For example, the area of a rectangle is $height \times width$, and the area of a pararrelogram is also the same. For different kinds of rectangles or pararrelograms, their height and width need not to be the same!

So in mathematics, if some formulas is provided, change (substitute) the variables if needed. It will help you to understand the questions.

Example 1.3.2 *Given a formula* $(x+y)(x-y) \equiv x^2 - y^2$*, find* $(u+v)(u-v)$ *if* $u = 3x^2y$ *and* $v = 7xy^2$*. You'll find that it so strange that you substitute* x *by* $3x^2y$*! Therefore we change the formula to* $(u+v)(u-v) \equiv u^2 - v^2$*, which x in the formula is equal to u in the question. We directly substitute u and v and get* $(u+v)(u-v) = u^2 - v^2 = 9x^4y^2 - 49x^2y^4$*.*

1.4 Factorizing

The reverse action of expanding is called factorizing. It turns a polynomial in form of $a_0x^0 + a_1x^1 + ... + a_nx^n$ into products of polynomials in lower degree.

There're several ways to factorize a polynomial, and we will introduce a few here, some will be discussed in later section.

Grouping: $ab + ac = a(b+c)$

Example 1.4.1 $2x + 2y = 2(x+y)$ *and* $x^2 + x = x(x+2)$*.*

Grouping can be performed repeatly to factorize the polynomial completely.

Example 1.4.2 $x^2+3x+2 = x^2+2x+x+2 = x(x+2)+(x+2) = (x+2)(x+1)$

Example 1.4.3 $x^3 + 6x^2 + 11x + 6 = x^3 + x^2 + 5x^2 + 5x + 6x + 6 = x^2(x+1) + 5x(x+1) + 6(x+1) = (x+1)(x^2+5x+6) = (x+1)(x^2+2x+3x+6) = (x+1)[x(x+2)+3(x+2)] = (x+1)(x+2)(x+3)$

Identity is something that both sides MUST be equal despite different values of the variables. Equality holds for all values of all variables. Note that polynomials before factorizing is identical to the product after factorizing.

We have some identity to help us to factorize polynomials.

Identity 1.4.4 $(x \pm y)^2 \equiv x^2 \pm 2xy + y^2$

The $\pm$ sign means that if one takes positive (plus) , then all $\pm$ become plus. If one takes negative (minus), then all become minus.

Identity 1.4.5 $(x+y)(x-y) \equiv x^2 - y^2$

Example 1.4.6 $36x^2 - 9 = 9(4x^2 - 1) = 9[(2x)^2 - 1^2] = 9(2x-1)(2x+1)$

Example 1.4.7 $x^2 - 6x + 9 = x^2 - 2(3)(x) + 3^2 = (x-3)^2$

Cross method is one of the most important way to factorize polynomials with degree of 2 (quadratic).

When we have a quadratic polynomial $x^2 + bx + c$, and we want to factorize it into form of $(x+d)(x+e)$. We expand $(x+d)(x+e)$ and get $x^2 + (d+e)x + de$. Since they're identical, we compare the coefficient and get $d+e = b$ and $de = c$. We can find d and e easily by trial and error if they're integers.

Similarly, when we have a quadratic polynomial $ax^2 + bx + c$, and we want to factorize it into form of $(dx+e)(fx+g)$. We expand $(dx+e)(fx+g)$ and get $dfx^2 + (dg+ef)x + eg$, we can also find d,e,f and g within integers.

Note that the factorization result is unique!

Example 1.4.8 $x^2 + 7x + 12 = (x+3)(x+4)$ *since* $3 + 4 = 7$ *and* $(3)(4) = 12$.

Example 1.4.9 $5x^2 + 17x + 6 = (5x+2)(x+3)$ *since* $(5)(1) = 5$, $(5)(3) + (2)(1) = 17$ *and* $(2)(3) = 6$.

1.5 Identity

Quadratic identity:

Identity 1.5.1 $(a+b+c)^2 \equiv a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

Cubic (degree 3) identity:

Identity 1.5.2 $(a \pm b)^3 \equiv a^3 \pm 3a^2b + 3ab^2 \pm b^3$

Identity 1.5.3 $a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$

$\mp$ means If $\pm$ takes positive, then it takes negative. If $\pm$ takes negative, then it takes positive.

Identity 1.5.4 $a^3 + b^3 + c^3 - 3abc \equiv (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Quadtic (degree 4) identity:

Identity 1.5.5 $(a \pm b)^4 \equiv a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

Others:

Identity 1.5.6 $a^n - 1 \equiv (a-1)(a^{n-1} + a^{n-2} + ... + a + 1)$

Identity 1.5.7 $a^n + 1 \equiv (a+1)(a^{n-1} - a^{n-2} + a^{n-3} - ... - a + 1)$ *If n IS ODD.*

Exercises

1. Show that the result of cross method is unique.
2. Factorize $a^4 - b^4$
3. Factorize $5x^2 - 13x + 6$.
4. Factorize $x^2z - y^2z$.
5. Factorize $x^3 + x^2y + xy^2 + y^3$.
6. Factorize $(x + y - 2xy)(x + y - 2) + (1 - xy)^2$. (very difficult!)
7. If $a^2 + a + 1 = 0$, find $a^8 + a^7 + 5$.
8. Expand $(x + y - 2)^2$.
9. (IMO prelim 09') Find the value of $\frac{1+2009^4+2010^4}{1+2009^2+2010^2}$.
10. Find the smallest possible value for k, which it's a postive integer satisfying $\frac{2008\times2010\times2012\times2014+k}{2011^4}$ is an integer.
11. If $m = x - y$ and $n = y - z$, express $x^2 + y^2 + z^2 - xy - yz - xz$ in terms of m and n.
12. Given a, b, c, d are non-negitive integers satisfying $ac+bd+bc+ad = 1997$. Find $a + b + c + d$.

Chapter 2

Function (I)

2.1 Introduction

Definition 2.1.1 *Function is a rule that maps a value to another value. For each input, the output is UNIQUE.*

It is a machine that you input numbers and it outputs numbers.We denote the function by $f(x)$, when x is our inputs.

We declare $f(x)$ equal to something to define a functino usually.

Definition 2.1.2 *If $f(x) = c$, where c is a constant, then $f(x)$ is a constant function. No matter how does x varies, $f(x)$ doesn't change.*

Definition 2.1.3 *If $f(x) = x^c$, where c is a non-zero constant. Then $f(x)$ is called a power function.*

Definition 2.1.4 *If $f(x)$ gives a polynomial of x, then it's a polynomial function.*

Example 2.1.5 $f(x) = 3x^2 - 9x + 4$ *is a polynomial function, and* $f(2) = 3(2)^2 - 9(2) + 4 = -2$.

Quick exercises:

1. Show that for the three types of function mentioned above, the output is unique for each input value.
2. If $f(x) = 7x^2 + 6x - 1$, determine whether it is a polynomial function, and find f(3).
3. If $f(x) = 7x^{99x} - 1$, determine whether it is a polynomial function, and find f(1) as well.
4. If $f(x) = 3x^2$, determine whether it is a (i) power function; (ii) polynomial function.

5. If $f(x) = \frac{1}{2}+x$, determine whether it is a (i)power function; (ii) polynomial function.
6. If $f(x)^2 = x$, determine whether it is a (i) power function; (ii) polynomial function.
7. If $f(x) = \frac{2x^2-1}{4x^3-7x+2}$, determine whether it is a polynomial function.
8. If $f(x) = \frac{x^2-1}{x-1}$, determine whether it is a polynomial function.

2.2 Analysis on a polynomial

Definition 2.2.1 *The degree function is defined as: You input a polynomial and it outputs the degree of the polynomial. We denote it by $deg(f)$, wehere f is the function we input.*

This function will be useful in later section.

Definition 2.2.2 *Leading coefficient of a polynomial $f(x)$ is the coefficient of $x^{deg(f)}$.*

Example 2.2.3 *Let $f(x) = 4x^2 - 2x + 1$, then $deg(f) = 2$. The leading coefficient of this polynomial is 2.*

Definition 2.2.4 *The sgn(x) is defined as: If x is non-negative, then sgn(x)=1. If x is negative, then sgn(x)=-1.*

Definition 2.2.5 *The absolute value function $|x|$ takes the magnitude of x. i.e., If x is negative, then $|x| = -x$; if it's non-negative, then $|x| = x$.*

Example 2.2.6 $sgn(-10) = 1$, *and* $|-10| = 10$.

Proposition 2.2.7 *If the $sgn($leading coefficient of $f(x)) = 1$, then exist a x such that for all $y \geq x$, $f(y) \geq 0$. Inversely if the sgn(leading coefficient of f(x))=-1, then exist a x such that for all $y \geq x$, $f(y) \geq 0$.*

Example 2.2.8 *Given two polynomial, $f(x)$ and $g(x)$ with postive leading coefficient, and $deg(f) > deg(g)$. If $g(0) > 0 > f(0)$, show that $g(x) = f(x)$ exist real solution(s).*

Solution: Let $h(x) = f(x) - g(x)$, then $deg(h) = deg(f)$ and the leading coefficient of $h(x)$ is positive as well. The solution of $g(x) = f(x)$ is equivalent to $f(x) - g(x) = 0$. Since $h(0) = f(0) - g(0) < 0$, and the leading coefficient is positive, therefore there exist a point such that $h(x) > 0$ (we can select some big-enough x such that $h(x) > 0$. By Intermediate Value Theorem, there exist a x such that $h(x) = 0$).

A much easier solution: consider the worst case: All terms in $g(x)$ contains postive coefficient and all terms except the leading terms of $f(x)$ contain a

negative coefficient. We find that no matter how big is the coefficient of terms with lower degrees, there exist a x such that $x^{deg(f)} > a_n x^n$ $(n < deg(f))$. Therefore there must exist a x such that $f(x) - g(x) = 0$.

This example is served for advanced players, though.

2.3 Composite function

Definition 2.3.1 *We define $(f \circ g)$ by $f(g(x))$.*

Example 2.3.2 *Decompose $(x+1)^k$ into polynomial functions and power functions.*

Solution: Let $f(x) = x + 1$ and $g(x) = x^k$. Then $(x + 1)^k = g \circ f$.

Example 2.3.3 *Decompose $(x^2)^k$.*

Solution: let $f(x) = x^k$, $g(x) = x^2$. Then $(x^2)^k = f \circ g$.

Decomposing functions may not be needed every time. In this case we can simply denote one function: $h(x) = x^{2k}$.

The signifigance of composing function is the we can solve a function's zeros (i.e., $f(x) = 0$), through breaking down the function into simpler functions, and solve them step by step.

Example 2.3.4 *Solve the equation $(x^2 - 1)^2 - 5(x^2 - 1) + 6 = 0$.*

By considering $f(x) = x^2 - 1$ and $g(x) = x^2 - 5x + 6$, the equation becomes $g \circ f = 0$.

Solving $g \circ f = 0$ gives $f = 2$ or $f = 3$. Solving $f(x) = 2$ gives $x = \pm\sqrt{3}$ and solving $f(x) = 3$ gives $x = \pm 2$.

2.4 Domain and co-domain

Definition 2.4.1 *Domain is that what will you input to the function. It is most likely all real number. If f(x) is not well defined (e.g. $\frac{0}{0}$, then x is not included by the domain.)*

Definition 2.4.2 *The meaning of not well defined means the indeterminate form. The most common types of indeterminate form include $\frac{c}{0}$, $\pm\infty$, 0^∞. If the output isn't real for some values of x, without specfied, that x is excluded in the domains also. For example, square rooting a negative number.*

Example 2.4.3 *Given $f(x) = 3x^3 - 1$, the domain is all real number since whatever you input, it will still give you the output.*

Example 2.4.4 *Given $f(x) = \frac{1}{4-x}$, the domain of $f(x)$ is all real numbers except 4 since when $x = 4$, it's not well defined.*

Example 2.4.5 *Given $f(x) = \sqrt{x}$, the domain of $f(x)$ is all non-negative numbers.*

Definition 2.4.6 *Co-domain is that when you put all domain into the $f(x)$, what's the range that it'll output.*

We will disscus this later since it's about minimal and maximal.

2.5 Sequences

Definition 2.5.1 *A sequence is in the form of $\{a_1, a_2, a_3, ...\}$, where a_n are some numbers.*

It is the discrete form of function. For example consider $f(x) = x$ and $a_n = n$, $f(x) = a_x$ for integers.

We can define sequences by $a_n = f(n)$, or simply write $\{a_1, a_2, ...\}$.

We have to integrate a function to calculate the area between $y = f(x)$ and x-axis, but calculating the sum of sequences is not so hard.

Definition 2.5.2 *Arithemetic progression is in the form of a, $a + d$, $a + 2d...$, where d is the difference. The sequence in the form of $\{a, a + d, a + 2d, ...\}$ is called a arithemetic sequence (A.S.).*

Definition 2.5.3 *Geometric progression is in the form of a, ar, $ar^2...$, where s is the difference. The sequence in the form of $\left\{a, ar, ar^2, as^3, ...\right\}$ is called a geometric sequence (G.S.).*

Definition 2.5.4 *Recursive sequence (R.S.) is that the n^{th} term (i.e., a_n) can be represented by the previous $(n - 1)$ terms.*

Example 2.5.5 *$\{1, 2, 3, 4, ...\}$ is an A.S. with difference 1; $\{2, 4, 8, 16, ...\}$ is a G.S. with ratio 2 and $\{1, 1, 2, 3, 5, ...\}$ is a R.S. wtih $a_n = a_{n-1} + a_{n-2}$.*

Quick exercise: Determine the type of the following sequences, and find the pattern (difference/ratio/how does the term related to previous term). If they're not the three types mentioned, state $f(x)$ for $a_n = f(x)$.

1. $\{1, 3, 5, 7...\}$
2. $\{1, 2, 4, 8...\}$
3. $\{1, 4, 9, ...\}$
4. $\left\{1, \frac{1}{2}, \frac{1}{4}, ...\right\}$
5. $\{10, -20, -50, ...\}$
6. $\left\{4, \frac{3}{4}, \frac{3}{16}, ...\right\}$

7. $\{1, 3, 4, 7...\}$

8. $\{9, 4, 6, 0, 12, ...\}$

Lemma 2.5.6 *The formula* $\frac{a_n - a_1}{n-1} = d$ *helps us to find one of them if the remaining three among last term, fist term, number of terms and difference is given.*

Proposition 2.5.7 *Sum of A.S.*$=\frac{n(a_1+a_n)}{2}$

Lemma 2.5.8 *The formula* $\sqrt[n-1]{\frac{a_n}{a_1}} = r$ *help us to find one of them if the remaining three among last term, first, number of terms and ratio is given.*

Proposition 2.5.9 *Sum of G.S.:* $a + ar + ar^2 + ... + ar^n = \frac{a(1-r^n)}{1-r}$. *If* $|r| < 1$, *then the sum to infinity*$=\frac{a}{1-r}$

Proof on Proposition 2.4.9: Multiply both sides by $1 - r$, group out the a, then L.H.S. becomes $a(1 - r)(1 + r + 1^2 + ... + r^n)$. By previous identity it is equal to $a(1 - r^n)$, which is equal to R.H.S., prove done for the first part.

For the second part of the claim, we know that when $|r| < 1$ and n (number ot terms) tends to infinity, $1 - r^n$ tends to 1. Therefore we have the above formula.

The calculation about recursive sequences will be discussed later.

Definition 2.5.10 *A typical type of recursive sequence, the Fibonacci Sequence is defined as:*

Given a_1 *and* a_2, *then* $a_n = a_{n-1} + a_{n-2}$ *for all* $n \geq 3$.

Example 2.5.11 $\{1, 1, 2, 3, 5, 8, ...\}$ *is a Fibonacci Sequence.*

Exercises

1. Sum to infinity for A.S. is meaningless. Why?

2. Sum to infinity for G.S. when $|r| \geq 1$ is meaningless. Why?

3. A claim says that when we're finding number of terms for A.S., $n = \frac{|a_n - a_1|}{d} + 1$, Why?

4. Find the sum of $1 + 11 + 21 + ... + 2011$ and $4 - 2 - 8 - ... - 2012$.

5. Find the sum of $1 + \frac{2}{5} + \frac{4}{25} + ...$ and $1 + \frac{3}{2} + \frac{9}{4} + ...$

6. If $1 + \frac{x}{S-4x} + \frac{x^2}{S} + ... = x + 1$, find S and x.

7. Find the domain of $f(x) = x^2$ and $f(x) = \frac{2+x}{2-x} + \frac{3+x}{3-x} + \frac{4+x}{4-x} + ...$ respectively.

8. Can $\{1, 2, 4, 8, 16, ...\}$ be a recursive sequence? $a_n = a_{n-1} + 2a_{n-2}$.

Chapter 3

Equation (I)

3.1 Introduction

In a equation, equality does not hold in all time. We find all solution such that equality holds for the value. This process is called solving equation.

For example, $2 + x = 5$ is a equation. Only when $x = 3$, equality holds.

Linear equation is the most easiest type of equation, in the form of $ax+b = c$. The general solution is $x = \frac{c-b}{a}$.

Now, we'll introduce a few theorems on solving different equations. The first one is factor theorem.

Theorem 3.1.1 *Factor theorem state that $f(a) = 0$ if and only if $f(x)$ can be factorized with a factor $(x - a)$.*

Proof: We prove this by contradiction. If $f(x)$ can be factorized into products of lower degree without the factor $(x - a)$, then when we substitute x by a, value of every product isn't zero. A product of non-zero value won't give a zero result. That makes a contradiction.

Theorem 3.1.2 *If a polynomial with degree n can be factorized into forms of $a(x - a_1)(x - a_2)...(x - a_n) = 0$, ($a_n$ are constants), then a_1, a_2, ..., a_n are the ONLY solutions to the equation. We call it the root of the equation.*

Proof: By Factor theorem, all a_n's are roots of $f(x) = 0$, and then we have to prove that there're no other roots by contradiction. Again if there're a root excluded by a_n, then we substitude that root (r) into $f(x)$. $f(r)$ can be expressed by some products, but the value of every product is non-zero! That makes a contradiction.

Example 3.1.3 $2x^2 - 6x + 4 = 2(x + 1)(x + 2)$, *therefore the roots of* $2x^2 - 6x + 4 = 0$ *is* $x = -1$ *and* $x = -2$

But in most of the case, we can't factorize them within integers, like $5x^2 - 9x - 4$, it's impossible for us to factorize it.

Another method of solving equations is to eliminate some terms.

Proposition 3.1.4 *For a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, substitude $x = y - \frac{a_{n-1}}{na_n}$ will eliminate the $a_{n-1}x^{n-1}$ term.*

We prove this by bionomial theorem which will be discussed later.

Consider $a_n x^n + a_{n-1}x^{n-1}$ only since only these two terms can produce y^{n-1}.

$$a_n x^n + a_{n-1}x^{n-1} = a_n(y - \frac{a_{n-1}}{na_n})^n + a_{n-1}(y - \frac{a_{n-1}}{na_n}{}^{n-1})$$
$$= a_n(y^n - \frac{a_{n-1}}{a_n}) + a_{n-1}y^{n-1} + ... = a_n x^n + ...$$

Therefore the y^{n-1} is eliminated.

Quick exercises:

1. Given a polynomial $7x^3 - 2x^2 + x - 12$, how can a substitude y such that the x term is eliminated?

2. We now introduce a new theorem, the Remainder theorem. When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is b, then $f(a) = b$. Prove this by considering the factorization of $f(x)$, and hence show that the Factor theorem is a special case of the Remainder Theorem.

3.2 Quadratic equation

Theorem 3.2.1 *The general formula of the general form of quadratic equation $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.*

How can we obtain the formula? We're going to discuss this in the following section.

Lemma 3.2.2 *If $x^2 = c$, where c is non-negitive, then $x = \pm\sqrt{c}$.*

The proof of the lemma is so obvious and left as exercise.

Method 1: We reach this by proposition 3.1.4. Substitude $x = y - \frac{b}{2a}$ yields

$$ax^2 + bx + c = a(y - \frac{b}{2a})^2 + b(y - \frac{b}{2a}) + c = ay^2 - by + \frac{b^2}{4a} + by - \frac{b^2}{2a} + c = 0$$
$$ay^2 = \frac{b^2}{2a} - \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a}$$
$$y^2 = \frac{b^2 - 4ac}{4a^2} \iff y = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Therefore $x = y - \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$.

Method 2: Completing square.

We can see that in the above method, we try to transform the equation until one unknown is left in the form of square. In some given equation, we often complete the square to solve the equation.

Example 3.2.3 *Solve the equation* $2x^2 - 3x + 1 = 0$.

Solution: $2x^2 - 3x + 1 = 2(x^2 - \frac{3x}{2} + \frac{1}{2}) = 2(x^2 - \frac{3x}{2} + \frac{9}{16} - \frac{1}{16}) = 2(x - \frac{3}{4})^2 - \frac{1}{8}$
Therefore, $(x - \frac{3}{4})^2 = \frac{1}{16}$, and $x - \frac{3}{4} = \pm\frac{1}{4}$, $x = \frac{3\pm 1}{4}$.
Therefore the solutions are $\frac{3+1}{4} = 1$ and $\frac{3-1}{4} = \frac{1}{2}$.
We now try to complete the square of the general form.

$$ax^2 + bx + c = a(x^2 + \frac{b}{a} + \frac{c}{a}) = a(x^2 + \frac{b}{a} + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a})$$
$$a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a} = 0$$
$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \iff x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Therefore $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 3.2.4 *Solve the equation* $5x^2 - x - 3 = 0$.

Solution: Substitude a, b, c by $5, -1, -3$ respectively.
We have $x = \frac{-(-1) \pm \sqrt{-1^2 - 4(3)(-5)}}{2(5)} = \frac{1 \pm 11}{10}$.

Exercises

1. Refer to the quick exercise. Eliminate the x term and try to gives the general formula. Is it the same?
2. Look at the general formula again. List some special cases that only gives ONE solution. Factorize those equation's polynomial and state you observations.
3. Refer to the general formula, list some special cases that can't give REAL solution. They can't be factorized. Why?
4. Add the roots together. State your observations.
5. Multiply the roots together. State your observations.
6. If we multiply both sides by 2 before using the formula or other methods, can we obtain the same result?
7. Solve the equation $3x^2 - 4x - 2 = 0$.
8. Solve the equation $11x^2 + 71x + 4 = 0$.
9. Show that if all coefficient in the equation is postive, then all roots are negative.

3.3 Sum and product

Theorem 3.3.1 *If r_1 and r_2 are roots of a quadratic equation $x^2+bx+c=0$, then $r_1+r_2=-b$ while $r_1r_2=c$.*

Proof: By Factor theorem, $(*) = x^2+bx+c = (x-r_1)(x-r_2)$. Expand the R.H.S. gives $x^2-(r_1+r_2)+r_1r_2 = (**)$. We compare $(*)$ and $(**)$ and the prove is done.

Alternative proof: For $x^2+bx+c=0$, by the general formula we have $x = \frac{-b\pm\sqrt{b^2-4c}}{2}$.

$(\frac{-b+\sqrt{b^2-4c}}{2}) + (\frac{-b-\sqrt{b^2-4c}}{2}) = \frac{-2b}{2a} = \frac{-b}{a}$

$(\frac{-b+\sqrt{b^2-4c}}{2})(\frac{-b-\sqrt{(-b^2)-4c}}{2}) = \frac{1}{4}(b^2-(\sqrt{b^2-4c})^2) = \frac{4c}{4} = c$

Now, consider a cubic equation $x^3+ax^2+bx+c=0(*)$ contains roots r_1, r_2, r_3. Then $x^3+ax^2+bx+c = (x-r_1)(x-r_2)(x-r_3) = (**)$. Expand (**) gives $x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_2r_3+r_3r_1)x-r_1r_2r_3=0$ compare this equation to (*) gives:

$$\begin{cases} r_1+r_2+r_3=-a \\ r_1r_2+r_2r_3+r_3r_1=b \\ r_1r_2r_3=-c \end{cases}$$

Now we generalize the above result.

Theorem 3.3.2 *Given a polynomial $x^n+a_{n-1}x^{n-1}+...+a_1x+a_0 = (x-r_1)(x-r_2)...(x-r_n)=0$, then we have*

$$\begin{cases} \sum r_i = -a_{n-1} \\ \sum r_ir_j = a_{n-2} \\ ...... \\ r_1r_2...r_n = (-1)^n a_0 \end{cases}$$

$\sum r_i$ means the sum of all combinations of root, $\sum r_ir_j$ means the sum of all combinations of products of any two roots, etc.

The above theorem is called the **Viete's theorem**.

Example 3.3.3 *Given that r_1 and r_2 are roots of $x^2-3x+4=0$, evaluate $r_1^3+r_2^3$.*

Solution: $r_1^3+r_2^3 = (r_1+r_2)(r_1^2-r_1r_2+r_2^2) = (r_1+r_2)((r_1+r_2)^2-3r_1r_2) = 3(3^2-(3)(4)) = -9$.

Definition 3.3.4 *For numbers $\{r_1, r_2, ...r_n\}$, $\sigma_1, \sigma_2, ...\sigma_n$ are called the elementary function of $\{r_1, r_2, ...r_n\}$. $\sigma_1 = \sum r_i$ and $\sigma_2 = \sum r_ir_j$, etc.*

Definition 3.3.5 *Consider a function $f(x_1, x_2, ..., x_n)$. We put n values into it. If $f(x_1, x_2, ..., x_n) = f(x_2, x_3, ..., x_n, x_1) = ... = f(x_n, x_1, ..., x_{n-1})$, then the function is called cyclic.*

Example 3.3.6 $r_1^3+r_2^3+r_3^3$ *is cyclic since* $r_1^3+r_2^3+r_3^3 = r_2^3+r_3^3+r_3^1 = r_3^3+r_3^1+r_2^3$.

Example 3.3.7 *$r_1 - r_1r_2$ isn't cyclic since $r_1 - r_1r_2 \neq r_2 - r_2r_1$*

Theorem 3.3.8 *All cyclic function can be expressed in products and sums of elementary function.*

The proof is omitted.

Example 3.3.9 *Express $r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3$ by elementary function.*

Solution: By identity 1.5.4, it equal to $(r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_2r_3 - r_3r_1)(*)$

Also, $r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2r_1r_2 - 2r_2r_3 - 2r_3r_1$

Therefore $(*) = (\sigma_1)(\sigma_1^2 - 3\sigma_2)$.

3.4 Discriminant

Definition 3.4.1 *For the equation $ax^2 + bx + c = 0$, it's discriminant is $\Delta = b^2 - 4ac$.*

Theorem 3.4.2 *If $\Delta > 0$, then $ax^2 + bx + c = 0$ has TWO distinct REAL solution.*

If $\Delta = 0$, then $ax^2 + bx + c = 0$ has ONE repeated solution.

If $\Delta < 0$, then $ax^2 + bx + c = 0$ has NO REAL solution.

Proof: By the definition of Δ, the general formula becomes $x = \frac{-b\pm\sqrt{\Delta}}{2a}$. When $\Delta > 0$, then $\sqrt{\Delta} - b$ and $-\sqrt{\Delta} - b$ is distinct, and thus it has two distinct real solution.

When $\Delta = 0$, then the term $\pm\sqrt{\Delta}$ disappear, and then since there are no $\pm$ sign the the equation, it has only one real solution.

When $\Delta < 0$, then $\sqrt{\Delta}$ isn't a real number. Therefore it has no real solution. The meaning of no real solution will be discussed later.

Now we have to prove that when $\Delta = 0$, the solution is repeated.

$$\Delta = b^2 - 4ac = 0 \iff b^2 = 4ac \iff \frac{b^2}{4a} = c$$
$$ax^2 + bx + c = a(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}) = a(x - \frac{b}{2a})^2$$

By factor theorem, it has one repeated root.

3.5 Simultaneous equation

Definition 3.5.1 *Simultaneous equation means there're more than one variables in the equation sets.*

Usually number of variables and number of equations should be the same.

Example 3.5.2 *Solve the equation* $\begin{cases} 2x + y = 7 & \text{(1)} \\ 4x - y = 5 & \text{(2)} \end{cases}$

The (1) and (2) denotes the first and second equation respectively. Threr're two methods to solve the equation.

Method 1: Substitution method. It means that you find the relationship between some variables, and substitude them to eliminate a variable.

Solution: From (1), $y = 7-2x$ (3). Put (3) into (2): $4x-y = 4x-(7-2x) = 6x - 7 = 5$, $x = 2$. Put $x = 2$ into (3) yields $y = 7 - 2(2) = 3$.

Method 2: Elimination method: It means that by adding between equations, you eliminate some terms. If there are only one variable left, then we can solve it easily.

Solution: (1)+(2)= $(2x+y)+(4x-y) = 6x = 5+7 = 12$, $x = 2$. Put $x = 2$ into (1) gives $2(2) + y = 7$, $y = 3$.

Example 3.5.3 *Solve the equation* $\begin{cases} ab + a = 8 & \textit{(1)} \\ ab + b = 9 & \textit{(2)} \end{cases}$

Solution: (2)−(1): $(ab + b) - (ab + a) = b - a = 9 - 8 = 1 \iff b = 1 + a$ (3). Put (3) into (1): $a(a + 1) + a = 8 \iff a^2 + 2a - 8 = 0$, $a = 2$ or $a = -4$.

When $a = 2$, $b = 3$. When $a = -4$, $b = -3$.

We say that the solutions of the simultaneous equation is $(a, b) = (2, 3)$ and $(-4, -3)$.

Exercises

1. By Viete's theorem or otherwise, state the relationship between roots and coefficients of equation in the fourth degree.
2. If some of the roots is repeated, does the Viete's theorem still holds? Explain.
3. A student claim that $a^2b+b^2c+c^2a$ isn't cyclic because $a^2b+b^2c+c^2a \neq b^2a+a^2c+c^2b$ (a and b is interchanged). Determine whether the claim is correct or wrong and explain.
4. If a, b, c are roots of $5x^3 - 6x + 4x - 1 = 0$, evaluate $a^3 + b^3 + c^3 + 3abc$.
5. If x and y are roots of $t^2 - 20x + 18 = 0$, find $x^3 + 20y^2 - 18x$.
6. $a^a + a^b + b^a + b^b$ is cyclic. Can it be expressed in terms of elementary function? explain.
7. Refer to question number 4. If both a and b are odd numbers, disprove the above claim.
8. Express $a^4 + 3a^3b + 5a^2b^2 + 3ab^3 + b^4$ in terms of elementary functions.
9. When $\Delta < 0$, it can't be factorized. Why?
10. Why does quadratic simultaneous equations also give two sets of solution?

11. Solve the equation $\begin{cases} a+b+c=0 \\ 3a-5b=c+4 \\ 5c+b=2a-1 \end{cases}$.

12. Solve the equation $\begin{cases} 6u+v=29 \\ 2v-u=4 \end{cases}$.

13. Solve the equation $\begin{cases} u^2-4v=29 \\ v^2-3u=4 \end{cases}$.

14. Solve the equation $t^3-7t+6=0$ by Factor theorem or otherwise, and hence solve $\begin{cases} a+b+c=0 \\ ab+ac+bc=-7 \\ abc=-6 \end{cases}$.

15. Solve the equation $\begin{cases} x+y=9 \\ x^2+y^2=2x^2+3 \\ y^2+z^2=30 \end{cases}$.

16. (IMO prelim 94) Solve the equation $\begin{cases} x+y=13 \\ y^2+z^2-yz=25 \\ x^2+z^2+xz=144 \end{cases}$.

17. Solve the equation $\begin{cases} a^2+b-c=38 \\ b^2-3c^4=a \\ c^4+b=a-2 \end{cases}$.

Chapter 4

Logarithm function

4.1 Revisit: Indice Law

Let's review the six indice law we have introduced.

1. $x^{a+b} = (x^a)(x^b)$
2. $x^{a-b} = \frac{x^a}{x^b}$
3. $(x^a)^b = x^{ab}$
4. $x^{-a} = \frac{1}{x^a}$
5. $x^0 = 1$
6. $x^{\frac{b}{a}} = \sqrt[a]{x^b}$

Have you noticed some patterns? You can see that in the indice laws, the plus or minus sign in the power becomes muliplication or division. And the multiplication on the powers becomes a double power. Now, we will introduce a function that take away the base.

Definition 4.1.1 *We define the logarithm functino as:*
If $a^x = b$, where a is positive (then b must be positive), then $log_a b = x$.
If $a = 10$, then we have a habit to write it as $logx$ usually.

Example 4.1.2 *Since $2^3 = 8$, therefore $log_2 8 = 3$.*

Definition 4.1.3 *If $f(x) = log_a x$, then $f(x)$ is called a logarithm function.*

Proposition 4.1.4 *The main principle of logarithm is :*
$log(ab) = log(a) + log(b)$.
We take log (apply the functino to both sides) to both sides of the indice laws above to get the log version of indice laws.

Quick exercises:

1. Evaluate $log1000$.
2. Taking log to a negative numbers is impossible. Why?
3. Show that $log_c(\frac{a}{b}) = log_c a - log_c b$
4. Show that $log_c x^a = alog_c x$
5. Refer to question number 2, there are some numbers x such that taking log twice to x is possible (i.e., $log(logx)$), find the smallest k such that for all $x > k$, $log(logx)$ is possible.

4.2 Logarithm function

Proposition 4.2.1 *Some important log version of indice law:*

- $loa(ab) = loga + logb$; $log(\frac{a}{b}) = loga - logb$;
- $log(a^b) = bloga$;
- $log(\frac{1}{a}) = -loga$;
- $log(1) = 0$;
- $log_a x = \frac{log_b x}{log_b a}$

Note that the all is suitable for any base except the last law, which the base is specfied. Proofs of the laws should be done in the previous exercises, while we will perform the last one here.

Proof: We first prove that $log_a x = \frac{logx}{loga}$.

Note that $10^{logk} = k$ by definition. Let $b = log_a x = \frac{logx}{loga}$. Then

(1): $b = log_a x \iff a^b = x$

(2): $b = \frac{logx}{loga} \iff bloga = logx \iff 10^{bloga} = (10^{loga})^b = a^b = x$

We are done by comparing (1) and (2).

Next, by the above claim, we have $\frac{log_b x}{log_b a} = \frac{\frac{logx}{logb}}{\frac{loga}{logb}} = \frac{logx}{loga} = log_a x$.

Example 4.2.2 *Evalute $log35 - log7 + log2$.*

Solution: $log35 - log7 + log2 = log(\frac{(35)(2)}{7}) = log10 = 1$.

Example 4.2.3 *Given $log2 = a$, $log3 = b$, evalute $log225$ and $log1920$.*

Solution: $log225 = log((3^2)(5^2)) = 2log3 + 2log5 = 2b + 2log5 + 2log2 - 2log2 = 2b - 2a + 2log(5 \times 2) = 2b - 2a + 2log10 = 2b - 2a + 2$.

$log1920 = log(10 \times 2^6 \times 3) = 1 + 6log2 + log3 = 6a + b + 1$.

Example 4.2.4 *i) Solve the equation* $z^2 + (1-b)z - b = 0$

ii) By (i), solve $\begin{cases} \frac{x}{y} = 5 \\ x^{\log y} = 2 \end{cases}$

Solution: (i) $z^2 + (1-b)z - b = z(z+1) - b(z+1) = (z-b)(z+1) = 0$

$$z = b \text{ or } z = -1$$

(ii)By (1), $x = 5y$ (3). Put (3) into (2): $5y^{\log y} = 2$
Take log on both sides: $\log(5y^{\log y}) = \log y \log 5y = \log y(\log y + \log 5) = \log 2$

$$(\log y)^2 + \log 5 \log y - \log 2 = (\log y)^2 + \log(\frac{10}{2}) \log y - \log 2 = 0$$

$$(\log y)^2 + (1 - \log 2) \log y - \log 2 = 0$$

$$(\log y - \log 2)(\log y + 1) = 0$$

$$\log y = \log 2 \text{ or } \log y = -1$$

Therefore $y = 2$ and $x = 5y = 10$ or $y = \frac{1}{10}$ and $x = 5y = \frac{1}{2}$.

Example 4.2.5 *Simplify* $a^{\frac{\log(\log a)}{\log a}}$.

Let $x = a^{\frac{\log(\log a)}{\log a}}$, then $\log x = \log a^{\frac{\log(\log a)}{\log a}}$.
Then $\log a^{\frac{\log(\log a)}{\log a}} = \frac{\log(\log a)}{\log a}(\log a) = \log\log a$.
Since $\log x = \log\log a$, $x = \log a$.

4.3 Simple limit and differentiation

Definition 4.3.1 *The formal definition of limit (The* ϵ*-*δ *definition)*

For a function which maps reals to reals ($f : \mathbb{R} \to \mathbb{R}$)*, for constant* p *and* L*, we say the limit of* f *as* x *approaches* p *is* L $\lim_{x\to p} f(x) = L$ *if and only if for every* $\epsilon > 0$ *there exist a real* $\delta > 0$ *such that* $0 < |x - p| < \epsilon$ *implies that* $0 < |f(p) - L| < \delta$. *Note that the limit and* $f(p)$ *are independent.*

Definition 4.3.2 *A simplier version of the definition of limit:*

When a function $f(x)$ *is approching to* p*, the limit* $\lim_{x\to p} f(x) = L$ *is used to describe the tendency when* x *is approching to* p *in* $f(x)$.

Definition 4.3.3 *One sided limit:*

$\lim_{x\to p^+} f(x)$ *means the limit when* x *approches to* p *from the side which* $x > p$.
$\lim_{x\to p^-} f(x)$ *means the limit when* x *approches to* p *from the side which* $x < p$.
$\lim_{x\to p} f(x)$ *exists if and only if* $\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x)$.

Definition 4.3.4 *We say $f(x)$ is continuous at p if and only if $\lim_{x \to p^+} f(x) = f(p) = \lim_{x \to p^-} f(x)$.*

Proposition 4.3.5 $\lim_{x \to p^-} f(x) = \lim_{\epsilon \to 0^-} f(x + \epsilon)$, $\lim_{x \to p^+} f(x) = \lim_{\epsilon \to 0^+} f(x + \epsilon)$

Proof: Left for exercise.

The concept of continuous is extremely importnat in the limit and differentiation.

From the above definition we can conclude that $f(p)$ is equal to the limit approaching to p if it's continuous at p. It is given that all polynomials are continuous at every point.

Proof: Let $f(x) = \sum_{i=0}^{n} a_i x^i$. Then $\lim_{y \to x^+} f(y) = \lim_{y \to x^+} \sum_{i=0}^{n} a_i y^i = \lim_{\epsilon \to 0^+} \sum_{i=0}^{n} a_i (y + \epsilon)^i = \sum_{i=0}^{n} a_i x^i = f(x)$

The second part ($\lim_{y \to x^-} f(y)$) can be done similarly. Therefore it's continuous at every point.

Example 4.3.6 $\lim_{x \to 2} x + 2 = 4$

Example 4.3.7 $\lim_{x \to -1} x^{101} - 27x + 4 = 30$

There are some functions which are not continuous in every point.

Example 4.3.8 $\begin{cases} x + 3 & \textit{when } x > 3 \\ x - 2 & \textit{when } x \leq 3 \end{cases}$

In this example, $f(3)$ and both of the one-sided limit exists, but the two one-sided limit isn't equal, so it's not continuous at $f(3)$.

Now let's have a quick revision on the slope.

For a constant function, it's slope is 0.

For a line $y = ax + b$, it's slope is x while the y-intercept is b. But how can you find the slope of a curve? It's changing all the time. Leibniz (or Newton, whatever) developed a system called **calculus**, including differentiation and integration. Differentiation is used to find the slope of a curve, while integration is used to solve the area enclosed by a curve. We will only discuss differentiation here.

Definition 4.3.9 *A differentiation of a function $f(x)$, which can be expressed by $\frac{df}{dx}$, $f'(x)$, or even simply f' is defined as:*

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$f'(p)$ tells us the slope of $f(x)$ at p.

Example 4.3.10 *Let* $f(x) = 4x$*, then*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{4(x+h)-4(x)}{h} = 4$$

.

We can see that even stright line obeys the differentiation.

Example 4.3.11 *Let* $f(x) = x^2$*, then* $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}$

$$= \lim_{h\to 0} 2x + h = 2x$$

Proposition 4.3.12 *Some mentionable properties of limit:*
$\lim_{x\to p}(f(x) \pm g(x)) = \lim_{x\to p} f(x) \pm \lim_{x\to p} g(x)$
$\lim_{x\to p}(f(x)g(x)) = (\lim_{x\to p} f(x))(\lim_{x\to p}(g(x)))$
$\lim_{x\to p}\frac{f(x)}{g(x)} = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)}$

Theorem 4.3.13 *The product rule states that:*
$(fg)' = f'g + fg'$

Prove: By definition.

$$(fg)' = \lim_{h\to 0}(\frac{f(x+h)g(x+h) - f(x)g(x)}{h})$$

$$= \lim_{h\to 0}\frac{(f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x))}{h}$$

$$= \lim_{h\to 0}(f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}) = f'g + fg'$$

Example 4.3.14 *Let* c *be a constant.* $(cx)' = (c)'(x) + (c)(x)' = 0 + c = c.$

Example 4.3.15 $(x^2)' = (x)'(x) + (x)(x)' = 2x.$

A special case of product rule claims the power rule:

Theorem 4.3.16 *Power rule states that* $(x^n)' = nx^{n-1}$*.*

Proof by mathematical induction.
Firstly when $n = 1$, it's obviously correct. (As shown in the example above) Then let $n = k$ is correct. (i.e., $(x^k)' = kx^{k-1}$) Then $(x^{k+1})' = (x)(x^k)' + (x)'(x^k) = kx^k + x^k = (k+1)x^k$. By mathematial induction, it's correct for all positive integers.
In fact, it's suitable for all real numbers k.

Theorem 4.3.17 *The quotient rule claims that* $(\frac{g}{h})' = \frac{g'h - gh'}{h^2}$*.*

The proof is complex and omitted here.

A special case of quotient rule is the reciprocal rule:

Theorem 4.3.18 *The reciprocal rule claims that* $(\frac{1}{g})' = \frac{-g'}{g^2}$.

Note that $g(x) \neq 0$.

Theorem 4.3.19 *The Chain rule states that:* $(f \circ g)' = f' \circ g + g'$.

Proof: omitted.

Quick exercises:

1. Show that $(cf)' = cf'$.
2. Show that $(f+g)' = f' + g'$.
3. Show that $(fgh)' = f'gh + fg'h + fgh'$.
4. Evaluate $(4x^2 - 7x + 2)'$.
5. Evaluate $(\frac{3x^2-5x}{x+1})'$.
6. Evaluate $(\frac{x^\pi}{3.14})'$.

4.4 Natrual logarithm

Definition 4.4.1 *e, which is a mathematical constant, is defined as* $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. $e \approx 2.718281828\ldots$

We will discuss about this e in the next chapter.

Definition 4.4.2 *The natrual logarithm,* $\ln x = \log_e x$.

4.5 Problems

Example 4.5.1 *Evaluate* $\log(\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}})$.

Solution: By the identity $(a+b)^2 = a^2 + 2ab + b^2$, we change the form of identity to : $\sqrt{a} + \sqrt{b} = \sqrt{a+b+2\sqrt{ab}}$.

Now substitude $\sqrt{3 \pm \sqrt{5}}$ into it: $\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}} =$
$\sqrt{3+\sqrt{5}+3-\sqrt{5}+\sqrt{(3+\sqrt{5})(3-\sqrt{5})}} = \sqrt{6+2\sqrt{4}} = 10^{0.5}$.

Therefore it equals to 0.5.

Example 4.5.2 *Solve the equation* $2^k + k^2 = 100$.

Solution: Take log on both sides: $\log(2^k + k^2) = 2$, but we can't simplify.

In fact we do not need to use log. We put k^2 to another side and take log: $2^k = (10-k)(10+k)$. So $(10-k)$ and $(10+k)$ are both powers of 2. Consider $k < 10$ gives an easy trial, which $k = 6$.

(Unless specfied, treat e as a mathematical constant.)

1. Solve the equation $k^2 - 2^k = 1$, where k is an integer.
2. Solve the equation $x^y = y^x$, where they're distinct integers.
3. Solve the equation $\log(3x^2 + 8x + 6) = 0$.
4. Solve the equation $\log(x^2 + 7) = \log(4x + 3)$
5. Differentiate $\frac{x^3 - x + 1}{5x^5 - 4x^4 + 2x^2 - x + 10}$.
6. Try to prove the quotient rule by definition.
7. Try to prove the reciprocal rule by quotient rule.
8. Express $\ln x$ in terms of $\log x$.
9. Show that all indice law is applicable to natrual logarithm.
10. Refer to question one, evaluate $k^{\frac{\ln \ln k}{\ln k}}$
11. Evaluate $e^{\ln e^{10}}$.

Chapter 5

Complex numbers

5.1 Number sets

Number is divided in some catagories.

1. The natrual numbers or positive integers: $\{1, 2, 3, 4, ...\}$.
2. The integers: $\{..., -2, -1, 0, 1, 2, ...\}$
3. The rational numbers, which can be expressed in $\frac{a}{b}$, where a and b are integers. for example, $\frac{1}{2}$, $\frac{29}{7}$
4. The real numbers, for example, π, $\sqrt{2}$.
5. The irrational numbers, which mean it isn't rational but it is real. for example, $\sqrt{29}$.
6. The algebraic numbers, it include all numbers that CAN be roots of a polynomial $f(x) = 0$, where the polynomial contains integer coefficients only. It'll not be discussed here in deatail.
7. The transcentendal numbers, which means that it's not algebraic but real.
8. The last one is...

Each sets of numbers can have it representation. We write a integers simply by a number, for example, 5. Integers are also rational numbers. It can be written in the forms of rational number. For example, $\frac{5}{1} = 5$ is actually an integer, but it can also be a rational number. Some of the above catagories is under another catagory.

5.2 Complex number

Definition 5.2.1 *$i^2 = -1$, where i is called the imaginary number. All its arithemetic refers to the basic rules.*

Definition 5.2.2 *A complex number is in form of $a+bi$, where a and b are real numbers.*

You can treat i as a variable which you can substitude $i^2 = -1$.
Arithemetic of complex numbers:

Proposition 5.2.3 $(a + bi) + (c + di) = (a + c) + (b + d)i$
$(a + bi)(c + di) = ac + (ad + bc)i + bdi^2 = (ac - bd) + (ad + bc)i$

Before we come to the division of complex number, let's look at some notation first.

Definition 5.2.4 *$|z|$ is the absolute value where z is a complex number. $|z| = |a + bi| = \sqrt{a^2 + b^2}$.*
$\overline{z}$ is called the conjugate of a complex number. $\overline{a + bi} = a - bi$.

Lemma 5.2.5 $z\overline{z} = |z|^2$

Proof: $(a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$.

Proposition 5.2.6 $\frac{a+bi}{c+di} = (\frac{a+bi}{c+di})(\frac{c-di}{c-di}) = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$
In another way, $\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$.

Quick exercise:

1. Does complex numbers include real numbers? Explain.
2. Does irrational numbers include transcendental numbers? Explain.
3. Refer to the definition of absolute value. This definition is also suitable for real number (i.e., $|x| = xsgn(x)$). Why?
4. Refer to question number 3, prove that $sgn(x) = \frac{|x|}{x}$, as well as $sgn(x) = \frac{x}{|x|}$.
5. Show that $i^3 = \frac{1}{i}$.
6. Evaluate $\sqrt{-9}$.
7. Show that $(\overline{z} + \overline{w}) = \overline{z + w}$.
8. Show that $|z||w| = |zw|$.
9. Evaluate $(z + w)\overline{(z - w)}$, where $z = 1 + i$ and $w = 4 - i$.
10. Evaluate $\frac{z^3-2iw^2}{2z^2-5w}$, where $z = 1 + i$ and $w = 4 - i$.

5.3 A quick preview on trigonometry

Consider a point $P(x_1, y_1)$ on a unit circle. It satisfies $x_1^2 + y_1^2 = 1$.

Point P, origin O and another point $Q(x_1, 0)$ forms a right-angled triangle.

For different angles between OP and x-axis (from x-aixs clockwisely to OP), the trigonometery function is defined as follows:

Definition 5.3.1 *Definition of trigonometry functions:*
Cosine: $cos\theta = x_1$
Sine: $sin\theta = y_1$
Tangent: $tan\theta = \frac{sin\theta}{cos\theta} = \frac{y_1}{x_1}$
Note that x_1 *and* y_1 *CAN BE negative in some* θ.

Definition 5.3.2 *Reciprocals of trigonometric functions:*
Secant: $sec\theta = \frac{1}{cos\theta}$
Cosecant: $csc\theta = \frac{1}{sin\theta}$
Cotangent: $cot\theta = \frac{1}{tan\theta}$

Definition 5.3.3 *One radian* (π) *is defined as 180 degrees.*

In geometry means, we have the following definition:

Definition 5.3.4 *For* $0 \leq \theta \leq \frac{\pi}{2}$, *we have the following definition:*
$sin\theta = \frac{opposite\ side}{hypotonuse\ side}$
$cos\theta = \frac{adjacent\ side}{hypotonues\ side}$
$tan\theta = \frac{opposite\ side}{adjacent\ side}$

Theorem 5.3.5 *By Pythagoras Theorem,* $sin^2\theta + cos^2\theta = 1$.

Now we try to find the relationship between different functions.

Consider a line $L : x = y$, angle between L and the x-axis is $45°$. Also consider the ΔPQO as mentioned above.

We now consider two new point $P'(y_1, x_1)$ and $Q'(0, y_1)$ by interchanging the values of x_1 and y_1. $\Delta P'Q'O$ is a right-angled triangle as well. Since $\Delta OQP \cong \Delta P'Q'O$, we have $\angle P'OQ' = \angle OPQ = 90° - \angle POQ$, by considering the trigonometric functions:

On ΔPQO:
$sin\theta = y_1$, $cos\theta = x_1$, $tan\theta = \frac{y_1}{x_1}$.
On $\Delta P'Q'O$:
$sin\theta = x_1$, $sin\theta = y_1$, $tan\theta = \frac{x_1}{x_1}$/
Therefore,

Theorem 5.3.6 *When* $\phi + \theta = 90°$, *we have the following identity:*
$sin\theta \equiv cos\phi$
$cos\theta \equiv sin\phi$
$tan\theta \equiv \frac{1}{tan\phi}$

Some special values:

When $\theta = 0°$, $sin\theta = 0$, $cos\theta = 1$. When $\theta = 30°$, $sin\theta = \frac{1}{2}$, $cos\theta = \frac{\sqrt{3}}{2}$. When $\theta = 45°$, $sin\theta = cos\theta = \frac{1}{\sqrt{2}}$.

Theorem 5.3.7 *Angle Bisector Theorem states that in ΔABC, when $\angle BAC$ is bisected by AD which D is on BC, $AB : AC = BD : CD$.*

The Proof is left for advanced readers. It can be done by basic definition of trigonometry and geometry skills.

Now we can proof the values of trigonometry functions when $\theta = 30°$ and $45°$.

Consider a triangle ΔABC, $AB = 2$, $BC = \sqrt{3}$, $AC = 1$. The angle bisector of $\angle BAC$ cuts BC at D. By Angle Bisector Theorem, $BD : BC = AB : BC = 2 : 1$, therefore $BD = \frac{2\sqrt{3}}{3} = \frac{2}{\sqrt{3}}$ and $CD = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$. By Pythagoras Theorem. $AD = \sqrt{1^2 + (\frac{1}{\sqrt{3}})^2} = \frac{2}{\sqrt{3}} = BD$. By the properties of isosceles triangle, $\angle ABD = \angle BAD = \angle DAC$ and therefore we have $3\theta = 90°$, and $\theta = 30°$.

Now we focus on the differntiation on $\sin x$ and $\cos x$.

Theorem 5.3.8 $(\sin x)' = \cos x$ *and* $(\cos x)' = \sin x$.

A famous method, the Taylor series is used to approximate the value of a function by a polynomial.

Theorem 5.3.9 $f(x) \approx \sum_{i=0}^{k} \frac{f^i(a)(x-a)^i}{i!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + ... + \frac{f^k(a)(x-a)^k}{k!}$

Maclaurin series is a simplified version of the Taylor series.

Theorem 5.3.10 $f(x) \approx \sum_{i=0}^{k} \frac{f^i(0)x^i}{i!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + ... + \frac{f^k(0)x^k}{k!}$

When k tends to infinity (sum to infinity), most likely both sides are equal.

That is, $f(x) = \sum_{i=0}^{\infty} \frac{f^i(0)x^i}{i!}$

Example 5.3.11 *We can get* $\sin x = \sum_{i=0}^{\infty} \frac{x^{2n+1}(-1)^{i+1}}{(2n+1)!}$ *by Maclaurin series.*

Note that $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^3(x) = -\cos x$ *and* $f^4 = \sin x = f(x)$, *this cycle is repeated every four times.*

Also, note that $\pm \sin 0 = 0$ *and* $\pm \cos 0 = \pm 1$.

$\sin x = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^3(1)x^3}{3!} + ...$

$= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{-x^7}{7!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

which is equal to that sum.

Exercises

1. Proof the angle bisector theorem.
2. Show that $sin\theta = cos\theta = \frac{1}{\sqrt{2}}$ when $\theta = 45°$.
3. Write a table on the six trigonometric functions (sine, cosine, tangent and their reciprocals) when $\theta = 0°, 30°, 45°, 60°, 90°$.
4. When $\theta = 0°$, $sin\theta = 0$. Why?
5. When $\theta = 90°$, $tan\theta$ is undefined. Why?
6. Show that $sin^2\theta + cos^2\theta = 1$ by geometric means.
7. Show that $sin\theta cos\theta(tan\theta + cot\theta) = 1$.
8. Simplify $(1 - tan\theta)(1 + cos\theta)(1 + tan\theta)(1 - cos\theta) + (1 - sin\theta)(1 + sin\theta)$.
9. Show that $tan\theta + cot\theta = \frac{1}{sin\theta cos\theta}$
10. Show that trigonometry functions are periodic, repeating every 2π.
11. Show that when $\theta = (45 + 90k)°$, $|sin\theta| = |cos\theta| = \frac{1}{\sqrt{2}}$.
12. Express $\cos x$ by Taylor series.
13. Differentiate $\sin x$'s Taylor series and compare it with $\cos x$.

5.4 The number e and Exponential function

Now we will learn about the mathematical constant e, as well as its application. It may not be useful in the current stage, but it invloves in many calculation of calculus, and hence the introdction of e should not be missed. Also, more historical content is given here.

Before we analysis on e, let's look at the calculation of interests:

Let t=time (in terms of one unit), I=interest with one unit of time and P be the original principle.

Simple interest: $P(1 + tI)$

Compound interest: $P(1 + I)^t$

You may find that when the unit of time is shortened, the compound interest increases. It based on a simple fact:

$(1 + I)^t \leq (1 + \frac{I}{k})^{kt}$ for $k \geq 1$.

However, it'll not tend to infinity. As mentioned, $e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$.

It also has different representations.

Definition 5.4.1 *If $f(x) = a^x$, where $a > 0$.*

Note that if $a = 1$, then it reduces to a constant function.

If $a > 1$, then $f(x)$ increses with x. If $a < 1$, then $f(x)$ decreses when x decreases.

Lemma 5.4.2 *For all positive $i > 1$, there exist a positive number k, such that $i^k = k^i$.*

The proof of lemma: left for exercise.

Lemma 5.4.3 *Another version of claim:*

For any $i, j > 1$, there exist infinitely many k such that $i^k > k^j$.

Taking log gives $\log(i^{\frac{1}{j}}) > \log(k^{\frac{1}{k}})$, it means that $\sqrt[j]{i} > \sqrt[k]{k}$.

We know that $\sqrt[k]{k}$ tends to 1 when k tends to infinity. At the same time $\sqrt[j]{i} > 1$ since $i, j > 1$. Therefore when k is big enough, $\sqrt[j]{i} > \sqrt[k]{k}$.

From the above lemma we can see that $f(x) = a^x$ is increasing monotonically and faster than pwoer functions and even

$$\frac{x}{dx}a^x = (\ln a)a^x$$

where a is a positive constants and ln is the natrual logarithm.

Example 5.4.4 $\frac{d(2^x)}{dx} = (\ln 2)2^x$.

By the formula we can find that $\frac{de^x}{dx} = \ln e e^x = e^x$.

One of the importance of the functino $f(x) = e^x$ is that after differentiation, it's still the same. The function $f(x) = e^x$ is called the **exponential function**. Furthermore, then function $f(x) = e^x$ is specfied as $\exp(x)$.

5.5 e and the complex number

Before we further define the number e, we shall introduce some new functions, as well as the convergence of a sequence.

Definition 5.5.1 *The factorial function x! is defined as: $x! = (1)(2)(3)...(x-1)(x)$, where x is a non-negative integer.*

It increases very fast, which is faster than an polynomials or exponentail functions. This proof is left for readers.

Now let's have a look on a sum:

Example 5.5.2 $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}...$.

It's called the harmonic series. You'll soon find that it becomes infinity when n tends to infinity. Why?

Proof:

$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + ...$

$\geq 1 + \frac{1}{2} + \frac{1}{2}...$ which obviously tends to infinity.

Definition 5.5.3 *C_r^n is defined as* $\frac{(n)(n-1)(n-r+1)}{(1)(2)...(r)} = \frac{n!}{r!(n-r)!}$.

It's very important and combinatorics, and a algebraic theorem is about it.

Theorem 5.5.4 *The binomial theorem states that:*
$(a+b)^n = \sum_{i=0}^{n} C_i^n a^i b^{n-i} = a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + ... + b^n$

Proof: we can prove this by combinatorical approach.

Firstly we define the idea of C_r^n in combinatorical approach. It's number of method to choose r items from n, where the order of the items isn't considered, i.e., choosing the first and second is same as choosing the second and the first.

Then, consider the coefficient of term $a^i b^{n-i}$. We choose i 'a'"'s from the $(a+b)$ term, and we have C_i^n methods to choose that. Then it's proved.

Now, consider $e = (1+\frac{1}{k})^k$, where k is extremely big integer.

By binomial theorem, $(1+\frac{1}{k})^k = 1 + C_1^k(\frac{1}{k}) + C_2^k(\frac{1}{k^2}) + ...(*)$

When k is very large such that $\frac{(k)(k-1)...(k-r+1)}{k^r}$ can be considered as 1, then (*) becomes $1 + \frac{1}{1!} + \frac{1}{2!} + ... = e$ and this is another expression of e^x.

In the other way, we find the Maclaurin series of e^x.

$e^0 = 1$, therefore $e^x = 1 + x + \frac{x^2}{2!} + ...$, you'll find that it's exactly the same!

Now, substitude i into e^x:

$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + ... = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + ...$

Now we group up the real and imaginary part: $e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!}) + i(x - \frac{x^3}{3!} + ...)$

You'll find that the both part is the Taylor series of $\sin x$ and $\cos x$!

Therefore we have one of the most famous formula, the Euler's formula:

Theorem 5.5.5 $e^{ix} = cisx = \cos x + i \sin x$.

Moreover substitute x by π gives $e^{\pi i} = \cos\pi + i\sin\pi = -1$! This is the most beautiful mathematical formula in the world.

Benjamin Peirce, a noted nineteenth century mathematician and Harvard professor, said, "*It is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth.*"

5.6 De Moivre's formula

By the above section, we have an important corollary:

Corollary 5.6.1 *Any complex number can be expressed as re^{ix}, where r is a constant.*

Proof: let $z = a + bi$. Consider $\frac{z}{|z|} = \frac{a}{|z|} + \frac{bi}{|z|}$, we have $\frac{z}{|z|}| = 1$.

Therefore $\frac{a}{|z|}$ and $\frac{b}{|z|}$ is a solution to $x^2 + y^2 = 1$.

Then, it can be subtitude by $\sin\theta$ and $\cos\theta$. $z = |z|(\frac{a}{|z|} + \frac{bi}{|z|}) = |z|cis\theta = |z|e^{i\theta}$.

Since this is an exponential function, we have $(cis(a))(cis(b)) = (e^a e^b) = e^{a+b} = cis(a+b)$.

Theorem 5.6.2 *A product of $cis(a_n)$ is equal to $cis(a_1 + ... + a_n)$.*

Theorem 5.6.3 *The standard version of De Moivre's theorem states that $(cis\theta)^n = cis(k\theta)$.*

Definition 5.6.4 *If $z = re^{i\theta}$, then $\arg(z) = \theta$.*

Since this functino is describing the size of angles, therefore $\arg(360° + x) = \arg(x)$.

Theorem 5.6.5 *If $\arg(z) = \alpha$ and $\arg(w) = \beta$, then $\arg(zw) = \alpha + \beta$*

Example 5.6.6 *If $w = \frac{1+\sqrt{3}}{2}$, evaluate w^{160}.*

We can find that $|w| = 1$ while $\arg(w) = 30°$, we have $w^{160} = 1^{160}cis(160(30°)) = cis120° = \frac{-1+\sqrt{3}i}{2}$

Example 5.6.7 *If $w = \frac{-1+\sqrt{3}}{2}$, evaluate $(w+1)^{20}$.*

Method I: $w + 1 = \frac{1+\sqrt{3}}{2} = cis30°$, $(w+1)^{20} = cis(20(30°)) = cis600° = cis120° = \frac{-1+\sqrt{3}}{2}$.

Method II: We can see that w is a root of $x^2+x+1 = 0$ as well as $x^3 - 1 = 0$. Then $(w+1)^{20} = (-w^2)^{20} = w^{40} = w$.

Exercise:

1. Prove lemma 5.4.2..
2. Differentiate $2^{(x^2)}$.
3. Differentiate x^x.
4. Determine $C_0^n + C_1^n + ... + C_n^n$.
5. Prove identity 1.5.5..
6. Evaluate $(1+i)(2+i)(3+i)$.
7. Show that $\arg((1+i)(2+i)(3+i)(4+i)...)$ does not converge.
8. A quadratic equation has a root $z = 2(e^{i\theta})$, find the general form of the quadratic equation.
9. If $x^2 - 19w + 25 = 0$ has root α and β, find equation that have root $\sqrt{\alpha}$ and $\sqrt{\beta}$.
10. If $w = -1 + \sqrt{3}i$, evaluate $(w+2)^{40}$.

5.7 Trigonmetry by the means of complex numbers

CAST rule gives the relationship between the functions for $180°\pm x$ and $360°\pm x$.

Proposition 5.7.1 *The case for* $180° - x$*: (S)*
$\sin(180° - x) = \sin x$, $\cos(180° - x) = -\cos x$ *and* $\tan(180° - x) = -\tan x$.

Proposition 5.7.2 *The case for* $(180° + x)$ *(T)*
$\sin(180° + x) = -\sin x$, $\cos(180° + x) = -\cos x$ *and* $\tan(180° + x) = \tan x$.

Proposition 5.7.3 *The case for* $(360° - x)$ *(C)*
$\sin(360° - x) = \sin(-x) = -\sin x$, $\cos(360° - x) = \cos(-x) = \cos x$ *and* $\tan(360° - x) = \tan(-x) = -\tan x$.

Proposition 5.7.4 *The case for* $(360° + x)$ *(A): equal after eliminating* $360°$.

Example 5.7.5 $\sin(270°-x) = \sin(180°+(90°-x)) = -\sin(90°-x) = -\cos x$.

Example 5.7.6 $\tan(720° + x) = \tan x$.

Now we try to obtain different trigonometric formulae through e^{ix}.

Example 5.7.7 *Deduce the formula* $\sin(2x) = 2\sin x\cos x$.

By considering $e^{2ix} = cis(2x) = \cos(2x) + i\sin(2x)$. At the same time, $e^{2ix} = (\cos x + i\sin x)^2 = (cos^2x - sin^2x) + 2i\sin x\cos x$. By comparing the real and imaginary part, we have the above result.

A more general case suggest that $\sin(x + y) = \sin x\cos y + \sin y\cos x$. The proof is left for readers.

Theorem 5.7.8 $\sin x = \frac{e^{ix}-e^{-ix}}{2}$ *and* $\cos x = \frac{e^{ix}+e^{-ix}}{2}$.

Firstly we have $e^{ix} = \cos x + i\sin x$. At the same time, $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$. Adding them together gives $2\cos x = e^{ix} + e^{-ix}$, and therefore $\cos x = \frac{e^{ix}+e^{-ix}}{2}$.

Similarly we can find that $\sin x = \frac{e^{ix}-e^{-ix}}{2}$.

Example 5.7.9 *Simplify* $\cos x\cos y$.

$\cos x\cos y = \frac{1}{4}(e^{ix} + e^{-ix})(e^{iy} + e^{-iy}) = \frac{1}{4}(e^{\pm i(x+y)} + e^{\pm i(x-y)}) = \frac{1}{2}(\cos(x + y) + \cos(x - y))$.

Example 5.7.10 *Simplify* $\cos x + \cos y$.

By the above example, $2\cos x\cos y = \cos(x + y) + \cos(x - y)$, substitude $x = a + b$ and $y = a - b$ gives $\cos a + \cos b = 2\cos(\frac{a+n}{2})\cos(\frac{a-b}{2})$.

Example 5.7.11 *Express* $\tan(x + y)$ *in terms of* $\tan x$ *and* $\tan y$.

Firstly we have $\cos(x+y) = \cos x \cos y - \sin x \sin y$, where the proof is left for readers.

$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y}$

By reducing both denominator and numerator by $\cos x \cos y$, we have: $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

Example 5.7.12 *Express* $\sin(\frac{x}{2})$ *in terms of* $\cos x$.

By $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$, we have $\frac{1-\cos x}{2} = \sin^2 \frac{x}{2}$. Therefore $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$. It takes plus or minus referring to different values of x.

Exercise:

1. Simplify $\sin(180° + x)\cos(180° - x)\tan(-x) + 1$.
2. Express $\sin x \sin y$ and $\sin x + \sin y$ in terms of $\sin x$ and $\sin y$.
3. Express $\tan x \tan y$ in terms of $\tan x$ and $\tan y$.
4. Express $\csc x \sec x$ in terms of $\sin x$ and $\cos x$.
5. Express $\sin 2x \sin 2y$ in terms of $\sin x$ and $\sin y$.
6. Express $\sin(nx)$, where n is a positive integer, in terms of $\sin x$.
7. Express $\tan(nx)$, where n is a positive integer, in terms of $\tan x$. You can express your answer with C_r^n.
8. Express $\cos^3(2x)$ in terms of $\cos x$.
9. Express the other trigonometric functions in temrs of e^{ix}.

Chapter 6

Function (II)

6.1 Absolute value function

Definition 6.1.1 $|x|$ *takes the magnitude of* x *for* x *is a real number.*

It implies that if $x \geq 0$, then $|x| = x$ while $|x| = -x$ if $x < 0$. Another expression is $\sqrt{x^2} = |x|$. In complex numbers we have $|a + bi| = \sqrt{a^2 + b^2}$, and it's also the same with the definition of $|x|$. When x is real, then $b = 0$.

Definition 6.1.2 $sgn(x)$ *takes the sign before* x *for* x *is a real number.*

Example 6.1.3 $|1| = 1$, $|-10| = 10$, $|0| = 0$, $sgn(7) = +1$, $sgn(-7) = -1$.

Lemma 6.1.4 $|x|sgn(x) = \frac{|x|}{sgn(x)} = x$.

Mathematical proof: The proof of $|x|sgn(x) = x$ is left for readers. Now we have $(sgn(x))^2 = 1$, then $sgn(x) = (sgn(x))^{-1}$.

Proof by word: when $|x|$ takes the magnitude and $sgn(x)$ takes the sign, combining them gives a complete x. Therefore $|x|sgn(x) = x$. In another way, $|x|$ reduced it's sign and $(sgn(x))^{-1}$ somehow reduces the sign once again, so a double-elimination gives the original x.

Example 6.1.5 *Solve the equation (1):* $|x| = c$, *where* $c > 0$*; (2):* $|c| = \log x$ *and (3):* $|x^2 + 1| = 2x$

(1): $x = \pm c$. (2): Since $\log x > 0$, we have $c = \log x$,, then $x = 10^c$. (3): When $x^2 + 1 \geq 0$, $x^2 - 2x + 1 = 0$, $x = 1$; when $x^2 + 1 < 0$, $x^2 + 2x + 1 = 0$, $x = -1$ but rejected since $(-1)^2 + 1 \geq 0$.

Example 6.1.6 *Solve the equation* $|x + 5| = 2$ *and* $|x - 5| = -2$.

For the second equation, it strictly gives NO solution since $|x|$ is strictly greater than 0.

Case 1: Let $x + 5 \geq 0$. Then $x + 5 = 2$ which gives $x = -3$.

Case 2: Let $x + 5 < 0$, Then $-x - 5 = 2$ which gives $x = -7$.

Example 6.1.7 *Solve the equation* $|x-1|+|x+2|=0$. *Solve the equation as well as RHS=1, 3 or 5.*

Since $|x| \geq 0$, we have $x-1=x+2=0$ which is impossible. Therefore it has no solution.

Firstly we determine the behavior of LHS. For $x \geq 1$, it equals to $2x+1>3$. When $x<-2$, it equals to $-2x-1>3$. When $1 \geq x \geq -2$, it equals to 3. (Why?) Therefore the least value of the function is 3, and when it equal to 1 it has no solution. It also explains why when it equals to zero, it has no solution. If it equals to 3, the solution is $1 \geq x \geq -2$. When it equals to 5, we obtain the solutino by solving $2x+1=5$ or $-2x-1=5$ which gives $x=2$ or $x=-3$.

Corollary 6.1.8 $|x|=0$ *implies* $x=0$.

Skill: Absolute value function usually reach its least value when the variable is eliminated each other, or the most term (with higher coefficient) is eliminated itself by reaching zero.

Example 6.1.9 *Solve the equation* $|x-1|+2|x-2|=2$.

Least value of LHS is 1 when $x=2$. When $x>2$, it equals to $3x-5>1$. When $2 \geq x \geq 1$, it equals to $3-x$. When $x<1$, it equals to $5-3x>2$. Then we solve $3x-5=2$ and $3-x=2$ to get $x=\frac{7}{2}$ or $x=1$.

Example 6.1.10 *Solve the equation* $|x+2|x+1||=0$.

By corollary 6.1.8, $|x+1|=\frac{-x}{2}$, which implies that $x<0$.
Case 1: $x+1 \geq \frac{-x}{2}$, then $x=-\frac{2}{3}$. Case 2: $x+1<\frac{x}{2}$. Then $x=-2$.

Example 6.1.11 *Find the least value of* $f(x)=|x-1|+|x-2|+...|x-10|$, *hence show that the least value of* $f(x)=|x-1|+|x-2|+...+|x-k|$ *is equal to* $\frac{k^2}{4}$ *if* k *is even.*

Firstly $f(x)$ is at the least value when the variable is eliminated each other. i.e., it's between 5 and 6. consider $x=5.5$, then it equals to $4.5+3.5+...+0.5+0.5+1.5+...4.5=1+3+5+7+9=25$. Similarly, we have the same result for the second statement.

Example 6.1.12 *Find sum of all possible real* x *for* $\begin{cases} |2x|=y+2 \\ |y^2-x|=9x+4 \end{cases}$

Case 1a: $x \geq 0$, $y^2 \geq x$. Then we have $\begin{cases} 2x=y+2 \\ y^2-x=9x+4 \end{cases}$. By substituting $y=2x-2$, we have $(2x-2)^2-x=4x^2-9x+4=9x+4$, then $\begin{cases} x=0 \\ y=2 \end{cases}$ or $\begin{cases} x=\frac{9}{2} \\ y=7 \end{cases}$. Note that both solution satisfies the assumption.

Case 1b: $x \geq 0$, $y^2 < x$. Then we have $\begin{cases} 2x = y + 2 \\ x - y^2 = 9x + 4 \end{cases}$. By substituting $y = 2x-2$, we have $x-(2x-2)^2 = 4x^2+9x-4 = 9x+4$, then $\begin{cases} x = \pm\sqrt{2} \\ y = -2 \pm \sqrt{2} \end{cases}$. However, the solution $x = -\sqrt{2} < 0$ does not satisfies our assumption, so there's only one solution in this case. (Readers can also try to show that another solution satisfies the assumption.)

Case 2a: $x < 0$, $y^2 \geq x$. Then we have $\begin{cases} -2x = y + 2 \\ y^2 - x = 9x + 4 \end{cases}$. By substituting $y = -2x - 2$, we have $(-2x - 2)^2 - x = 4x^2 + 7x + 4 = 9x + 4$, then $\begin{cases} x = \pm\frac{1}{\sqrt{2}} \\ y = -2 \mp \frac{1}{\sqrt{2}} \end{cases}$. However $x = \frac{1}{\sqrt{2}} > 0$ does not satisfies our assumption.

Case 2b: $x < 0$, $y^2 < x$. Then we have $\begin{cases} -2x = y + 2 \\ x - y^2 = 9x + 4 \end{cases}$. By substituting $y = -2x - 2$, we have $x - (-2x - 2)^2 = -4x^2 - 7x - 4 = 9x + 4$, then $\begin{cases} x = -2 \pm \sqrt{2} \\ y = 2 \mp 2\sqrt{2} \end{cases}$ However, $x < 0 < y^2$, so it gives no solution.

Therefore sum of all possible x$= 0 + \frac{9}{2} + \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{9+\sqrt{2}}{2}$.

6.2 Absolute value function involving complex number

Now consider the absolute value function for complex number:

$$|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$$

By the knowledge of coordinate geometry, the circle on a Cartesian plane is represented by $(x - a)^2 + (y - b)^2 = r^2$. Similarly, a circle on an Argand diagram can be represented by $|z| = r$.

Example 6.2.1 *Solve the equation $|z - 2| = |z - i|$ for z is a complex number.*

Let $z = a + bi$. $(|z - 2|)^2 = (|z - i|)^2 \iff (a - 2)^2 + b^2 = a^2 - 4a + 4 + b^2 = a^2 + (b - 1)^2 = a^2 + b^2 - 2b + 1$. Then we can eliminate the square term. By comparing the real and imaginary part: $-4a + 4 = 0$, $a = 1$; $0 = -2b + 1$, $b = \frac{1}{2}$. Therefore $z = 1 + \frac{i}{2}$. Of course, $|z| = 0$ implies $z = 0$ strictly.

Example 6.2.2 *Solve the equation $|z| = 2$.*

It's equivalent to $a^2 + b^2 = 4$, the **set** of solution is described as a circle on the complex plane (Argand diagram).

Example 6.2.3 *Show that the equation $3|z - 2i| = |z + 8 - 2i|$ is represented by a circle, and find its radius. On the Argand diagram, P is representing $-7 + 17i$ while Q is a free point on the circle represented by that equation. Find the longest distance of PQ. (HKAL 2008)*

Let $z = a + bi$. $3|a + (b-2)i| = |(a+8) + (b-2)i|$. By squaring both sides,

$$9a^2 + 9(b-2)^2 = (a+8)^2 + (b-2)^2$$

$$8a^2 - 16a - 64 + 8(b-2)^2 = 0 \iff a^2 - 2a + 1 + (b-2)^2 = (a-1)^2 + (b-2)^2$$

$$(|z - 1 - 2i|)^2 = 9$$

Therefore the circle has radius $\sqrt{9} = 3$ and center $1 + 2i$. Now we try to solve the second part of the problem. We can state that the PQ is at maximum when PQ passes through the center of the circle. Let $1 + 2i$ be point O and the tangents of PQ and the circle be T_1 and T_2. We can show our assumption logically:

- Ray PQ cuts the circle at two point, unless PQ is a tangent.
- Let thw two intersection point be Q_1 and Q_2 according to the intersection order. $PQ_2 \geq PQ_1$, equality holds when PQ is the tangent.
- We take PQ_2 as the (longest) distance. Length of PQ is symmetry along PO.
- Obviously the tangent is the shortest distance among PQ_2.
- Therefore the longest distance come from the longest distance of arcQT, which PQ covers the diameter of the circle.

Another approch to show that our claim shows the longest distance of PQ:

Consider Q is a free point on the circle of that equation. By the triangular inequality [1], $PO + OQ \geq OQ$ and equality holds only when the three points does not form a triangle, i.e., they're colinear. Then the longest PQ must be equal to $PO + OQ = PQ + 3$.

Therefore P, O and Q are colinear. We can reach that we considering the slope of PO and OQ is the same. Slope of PO$=\frac{17-2}{-7-1} = \frac{-15}{8}$. Let Q representing $a + bi$, we can find Q by setting the equation $\begin{cases} |z - 1 - 2i| = 3 \\ \frac{b-2}{a-1} = \frac{-15}{8} \end{cases}$

By the second equation $b - 2 = \frac{15-15a}{8}$, by the first equation we solve $a = \frac{-7}{17}$ or $\frac{41}{17}$. We get the second solution since it's further from P. Then $b = \frac{-11}{17}$ and so Q is at $\frac{41}{17} - \frac{11i}{17}$.

Checking gives: $(\frac{41}{17} - 1)^2 + (\frac{-11}{17} - 2)^2 = 9$.

Therefore longest $PQ = \sqrt{(-7 - \frac{41}{17})^2 + (17 + \frac{11}{17}^2)} = 20$. Checking $PQ = PO + 3$: $\sqrt{(-7-1)^2 + (17-2)^2} + 3 = 20$.

As you can see, the calculation about complex value function can be very hard as two unknown[2] and quadratic equation is natrually produced. It can be even more complicated if some terms are outside the function.

Now we can study the absolute value function on complex number in forms of r$cis\theta$.

[1] It refers to in any triangle, sum of any two sides is always larger than the remaining side.

[2] Refers to real and imaginary part

Corollary 6.2.4 $|rcis\theta| = r.$

Proof: Recall for any positive real, $|x| = x$ and $\arg(w) + \arg(z) = \arg(wz)$. Then $|r| = r$ and $\arg(cis\theta) = \theta$. Therefore $|r| = r$. Moreover $|cis\theta| = 1$ by $\sin^2 x + \cos^2 x = 1$. Therefore $|rcis\theta| = r(1) = r$.

Example 6.2.5 *Find* $|2^{-i}|$.

$$2^{-i} = e^{-i\ln 2} = (e^{\ln 2})e^{(i)(\frac{180^\circ}{\pi})} = e^{\ln 2} cis \frac{180^\circ}{\pi}$$

Therefore $|2^{-i}| = e^{\ln 2}$.
Exercise:

1. Find the least value of $f(x) = |x-1| + |x-2| + ...|x-101|$, hence show that the least value of $f(x) = |x-1| + |x-2| + ... + |x-k|$ is equal to $\frac{k^2-1}{4}$ if k is odd.
2. Solve the equation $|x-1| = |x+1|$.
3. Solve the euqation $|x^2-1| = |x-1|$.
4. Solve the equation $|x + |x - |x+1||| = 3$.
5. Solve the equation $2|x+1| + 3|x-1| = 3$.
6. Solve the equation $|x^2 - 3|x-1|| = 4$.
7. Solve the equation $|x^2 - 12x + 32| = 4$.
8. Solve the equation $|x^2 - 5x + 6| = 0$ and $|x|^2 - 5|x| + 6 = 0$. Compare your result.
9. Solve the equation $3|z-1| = |z+4+i|$ where z is a complex number.
10. Solve the equation $5z = |4z + 3i|$ where z is a complex number.
11. Show that $sgn(x)$ and $\arg(x)$ are similar in it's function. In fact, $sgn(x)$ is the unit vector of a real number on the real number line, and $sgn(x)$ is the *real version* of $\arg(z)$. However, $\arg(x)$ only gives the angle. Please state function of changing complex number to the unit vector on an Argand diagram. Hence link the $sgn(x)$ and $\arg(x)$ together.
12. Rewrite the function $f(x^{-3i}) = e^{\ln x^3}$.

6.3 Integer function

Definition 6.3.1 *$[x]$, defined as the integer function, takes the integer part of x, $x \geq [x] > x-1$ and it is unique.*

Definition 6.3.2 *$\{x\}$, defined as the decimal function, takes the decimal part of x, $1 > \{x\} \geq 0$ and it is unique.*

Theorem 6.3.3 *$x = [x] + \{x\}$.*

Proof of existance is base on the fact: for $x \geq k > x-1$, we can always pick a k such that it's integer.

Proof on uniqueness: *If $x = [x]_1 + \{x\}_1 = [x]_2 + \{x\}_2$, then $[x]_1 = [x]_2$ while $\{x\}_1 = \{x\}_2$*

If $[x]_1 \neq [x]_2$, then $[x]_1 = [x]_2 + n$, where n is an integer. In order to hold the equality, $\{x\}_1 = \{x\}_2 - n$. If $n \neq 0$, then obviously $\{x\}_1$ will not be in the range defined.

Another definition states that:

Definition 6.3.4 *For any real x satisfying $n+1 > x \geq n$ for n is an integer, we say $[x] = n$ and $\{x\} = x - n = x - [x]$.*

Example 6.3.5 *Solve the equation $3[x] + 1 = 2x$.*

Arrange the terms yields $[x] + 1 = 2\{x\}$, by considering the range of RHS, $[x]$ can only be -1 or 0.

If $[x] = -1$, then $\{x\} = 0$, then $x = -1$. If $[x] = 0$, then $\{x\} = \frac{1}{2}$ and $x = \frac{1}{2}$.

Example 6.3.6 *Find sum of all possible x from the equation $2\{x\}[x] + 4 = 3\{x\} + 3x$. (PC 2010)*

By transformaing the RHS we will have $[x](2\{x\} + 3) = 6x - 4$. Since $LHS < 5[x] \leq 5x$, equality holds only when x is near to zero. We bound x roughly from -2 to 4. Considering different case on integer part, we will get the answer $\frac{-1}{8}$, $\frac{2}{3}$ and $\frac{4}{5}$ respectively. Then the sum is $\frac{43}{24}$.

Example 6.3.7 *Solve the equation $201x + 9[x] = 2009$*

Arrange the terms gives: $201\{x\} = 2009 - 210[x]$. However you'll have be consider 201 case here and it's impossible. But from the equation you can find that $[x]$ must be equal to 9, Otherwise LHS can't hold the equality. Therefore $201\{x\} = 2009 - 210(9) = 119$ so that $\{x\} = \frac{119}{201}$ and $x = \frac{1928}{201}$.

A special properties of this type of function is that, it seldom relys on the traditional general solution of quadratic or cubic equation. Instead, grouping will be very useful here. The following example is rather complex.

Example 6.3.8 *Solve the equation $[x]^3 - 7\{x\}[x] - 5 = 0$.*

Grouping gives $[x]([x]^2 - 7\{x\}) = 5$. Since $[x]^2 - 7\{x\} < [x]^2 - 7$, when $|[x]| > 3$, LHS>RHS and the equality can't hold.(Readers can try to show this one.) Therefore we bounded that $[x]$ = -2, -1, 0, 1 or 2.

For $[x] = 2$, it equals to $2(4 - 7\{x\}) = 5 \iff 14\{x\} = 3$, then $x = \frac{27}{14}$.

For $[x] = 1$, it equals to $1 - 7\{x\} = 5 \iff 7\{x\} = -4$ which is rejected. When $[x] = 0$, it's also rejected by the same reason.

For $[x] = -1$, it equals to $-(1 - 7\{x\}) = 5 \iff 7\{x\} = 6$, then $x = \frac{-1}{7}$.

For $[x] = -2$, it equals to $-2(4 - 7\{x\}) = 5 \iff 14\{x\} = 13$, then $x = \frac{-15}{14}$.

Therefore the three solutions are $\frac{27}{14}$, $\frac{-1}{7}$ and $\frac{-15}{14}$.

Example 6.3.9 *Let* $x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{100}}$, *find* $[x]$.

By the inequality $\frac{1}{\sqrt{x}} > \frac{2}{\sqrt{x}+\sqrt{x+1}} = 2(\sqrt{x+1} - \sqrt{x})$, we have $x > 2(\sqrt{101} - 1) > 18$. But how to show that $[x] = 18$? We have to use another inequality $\frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x-1}}$.

Similarly by $\frac{1}{\sqrt{x}} < \frac{2}{\sqrt{x}+\sqrt{x-1}} = 2(\sqrt{x} - \sqrt{x-1})$, we have $x < 2(\sqrt{100} - 0) = 20$. However this bonding isn't tight enough for us to show that $[x] = 18$. Now we use a trick: $x - 1 < 2(\sqrt{100} - 1) = 18$, which implies that $x < 19$. Therefore $[x] = 18$.

Example 6.3.10 *If* $r = \sqrt{5}$, *evaluate* $\{r\}^2 + 2\{r\} + 2\sqrt{5}$.

$\sqrt{5} \approx 2.2...$, so $\{r\} = \sqrt{5} - 2$. Substitute into the formula: $\{r\}^2 + 2\{r\} + 2\sqrt{5} = 9 - 4\sqrt{5} + 2\sqrt{5} - 4 + 2\sqrt{5} = 1$.

For this type of question, transforming the square roots will be important.

Example 6.3.11 *Find* $\sqrt{7 - 2\sqrt{3}} + \sqrt{3}$.

You'll find that $(2 - \sqrt{3})^2 = 7 - 2\sqrt{3}$, so $\sqrt{7 - 2\sqrt{3}} + \sqrt{3} = 2 - \sqrt{3} + \sqrt{3} = 2$.

Note: The floor function $\lfloor x \rfloor$ is equal to $[x]$, while the celling function $\lceil x \rceil$ is equal to $[x] + 1$ if x isn't integer. If it's integer, then it equals to x.

Exercise:

1. Simiplify $\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}}$.
2. Let $r = \sqrt{5} + 1$, evaluate $r^{2011} - 2r^{2010} - 4r^{2009}$.
3. Let $x = \sqrt{11 - 6\sqrt{2}}$. Evaluate $x + 2\{x\}^{-1}$.
4. Simplify $\sqrt{(7 + 4\sqrt{3})^3} + \sqrt{(7 - 4\sqrt{3})^3}$.
5. Let $a = 1 + 2^{\frac{1}{2}} + 2^{\frac{2}{3}}$. Evaluate $3(a^{-1} + a^{-2}) + a^{-3}$.
6. Solve the equation $\frac{5}{2}\{x\} + [x]^2 = x^2$.
7. Solve the equation $[x]^2 - 20\{x\} = [x]$.
8. Solve the equation $[x]^3 + 2[x]^2\{x\} - 13 = 0$.
9. Solve the equation $[x]^4 - 7[x]^2\{x\} - [x] - 5 = 0$. (very difficult!)

6.4 Multivariate function

Definition 6.4.1 *A function that inputs more than one variable to give out a value is called a multivariate function.*

Addition, subtraction, multiplication and division are in fact multivariate function. e.g., $f(a,b) = a+b$ and $f(a,b) = ab$.

Example 6.4.2 $f(x,y) = 3x + 2y^2$ *is a multivariate function.*

In fact, multivariate function can be used to factorize polynomials that contain more than one variable.

Definition 6.4.3 *A multivariate function* $f(a_1, a_2, ...a_n)$ *is called cyclic if* $f(a_1, a_2, ...a_n) = f(a_2, a_3, ...a_n, a_1) = .. = f(a_n, a_1, ...a_{n-1})$.

Definition 6.4.4 *A multivariate function is called symmettical if the value doesn't change even two of the variables are interchanged.*

Definition 6.4.5 *A multivariate functino is called alternating or skew if the value is multiplied by -1 if two of the variables are interchanged.*

It's a bit like the cyclic polynomial, but we can describe all types of function here.

Example 6.4.6 $f(x,y,z) = xy + yz + xz$ *and* $f(x,y,z) = x + y + z$ *is cyclic.*

Example 6.4.7 $f(x,y,z) = xyz$ *and* $f(x,y,z) = x + y + z$ *is symmetrical.*

Example 6.4.8 $f(x,y) = x - y$ *is aternating.*

Theorem 6.4.9 *A cyclic function is a product of other cyclic function.*

Note that $(x-y)(y-z)(z-x)$ isn't cyclic for each bracket, but the product is cyclic.

Example 6.4.10 $a^3 + b^3 + c^3 - 3abc$ *is cyclic and for* $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, *each bracket is cyclic also.*

Example 6.4.11 *Factorize* $f(x,y,z) = x^4(y^2-z^2) + y^4(z^2-x^2) + z^4(x^2-y^2)$.

When $x = y$, $f(x,x,z)$ equals to zero. Therefore $(x-y)$ is a factor of it. Similarly $(x-y)(y-z)(z-x)$ is it's factor. Remember that it's composed by the cyclic function and it's degree is 4. Therefore there must be one more bracket of the first degree, i.e., $(x+y+z)$. Let $f(x,y,z) = k(x+y+z)(x-y)(y-z)(z-x)$ for some k. By comparing coefficient we can get $k = 1$ easily. Therefore,

$$f(x,y,z) = (x+y+z)(x-y)(y-z)(z-x)$$

Definition 6.4.12 *A cyclic sum is defined as:* $\sum\limits_{cyc} f(a_1, a_2, ...a_n) = f(a_1, a_2, ...a_n) + f(a_2, a_3, ...a_n, a_1) + ... + f(a_n, a_1, ...a_{n-1})$.

It's a well-known but not so official notation. We would like to define the cyclic product here as well.

Definition 6.4.13 $\prod\limits_{cyc} f = f(a_1, a_2, ...a_n) f(a_2, a_3, ...a_n, a_1)...f(a_n, a_1, ...a_{n-1})$.

Example 6.4.14 *Define* $f(x, y, z) = x^2(y - z)$*, then* $\sum\limits_{cyc} f = x^2(y - z) + y^2(z - x) + z^2(x - y)$*. You can also simply write* $\sum\limits_{cyc} x^2(y - z)$.

Corollary 6.4.15 *A cyclic polynomial of degree k can always be expressed in terms of sum of elementary function of degree k.*

A cyclic sum need not to make all variable appear in the function. For example, $\sum\limits_{cyc} f = x + y + z$, then $f(x, y, z) = x$. However, number of terms in the sum is equal to the number of variables.

Definition 6.4.16 *General cyclic factor of polynomail (degree k):* $\sum\limits_{i=1}^{k} c_i\sigma_i$*, where* σ_i *is the elementary function and* c_i *is the coefficient of each elementary function.*

Example 6.4.17 *Factorize* $\sum\limits_{cyc} x(y^4 - z^4)$.

Similar to the examples before, $\prod\limits_{cyc}(x - y)$ is one of the factor. Now the original polynomial is degree 5 but our factor only have degree 3. By theorem 6.4.9, we have a cyclic function of second degree: $\sum\limits_{cyc}(c_1x^2 + c_2xy)$. By comparing the coefficients, $c_1 = c_2 = 1$. Therefore $\sum\limits_{cyc} x(y^4 - z^4) = \prod\limits_{cyc}(x - y) \sum\limits_{cyc}(x^2 + xy)$.

Example 6.4.18 *Show that if* $a + b + c = 0$ *and* $ab + bc + ca + 3m = 0$*, then* $\prod\limits_{cyc} f(a, b, c)$ *will contain terms which index is multiple of three only where* $f(a, b, c) = x^2 + ax + m$*. Hence if it is given that* $x^2 - 2x + m$ *is a factor of* $x^6 + 16x^3 + 64$*, factorize the polynomial.*

Let $x^2 + m = y$. Then $\prod\limits_{cyc} f(a, b, c) = y^3 + (a+b+c)xy^2 + (ab+bc+ca)x^2y + abcx^3 = y^3 - 3mx^2y + abcx^3 = (x^2+m)(x^4 - mx^2 + m^2) + abcx^3 = x^6 + abcx^3 + m^3$. and it satisfies the requirement. Let $x^6 + abcx^3 + m^3 = (x^2 - 2x + 4)(x^2 + bx + 4)(x^2 + cx + 4) = x^6 + 16x^3 + 64$. Comparing the coefficients gives the answer $(x^2 - 2x + 4)^2(x^2 + 4x + 4) = (x^2 - 2x + 4)^2(x + 2)^2$.

Multivariate function may not give the rule directly sometimes. In olympiads mathematics, they may not even have an unique expressions.

Example 6.4.19 *For a function $f(h,k)$ where h and k is positive integers, obey the following rule: (1) $f(1,1)=2$; (2) $f(h,k)=f(h-1,k)+k$ for $h\geq 2$; (3) $f(h,k)=f(h,k-1)+h$ for $k\geq 2$. If it has a unique expression show it out, also evaluate $h(5,8)$.*

By repeating rule 2 and 3, we have $f(h,k)=f(1,k)+(h-1)k=2+(k-1)+k(h-1)=1+hk$. It's unique since $f(h,k)=f(h,k-1)+h=f(h-1,k-1)+h+k-1=f(h-1,k)+k$. Therefore $f(5,8)=41$.

To show that the function is unique, we have to show that the order of all rules are interchangable. For the previous example, rule 2 and 3 is interchangle while rule 1 is not applicable before the use of rule 2 and 3. So showing that 2 and 3 is interchangable implies that it's unique.

Example 6.4.20 *Given a function $f(x,y)$ (x,y is real number) obeys the following rule: (1) $f(0,0)=2$; (2) $f(x,y)=f(\frac{\pi}{2}-y,\frac{\pi}{2}-x)-\frac{\pi}{2}(x+y-\frac{\pi}{2})$ (3) $f(x,y)=f(y+\frac{\pi}{2},x-\frac{\pi}{2})+\frac{\pi}{2}(x-y-\frac{\pi}{2})$. Determine whether it is unique, and show that $f(x,y)=2(\sin x+\cos y)-xy$.*

(2)-(3): $f(\frac{\pi}{2}-y,\frac{\pi}{2}-x)-f(y+\frac{\pi}{2},x-\frac{\pi}{2})=f(y+\frac{\pi}{2},x-\frac{\pi}{2})=\pi(\frac{\pi}{2}-x)$ Now consider $(y+\frac{\pi}{2})(x-\frac{\pi}{2})-(\frac{\pi}{2}-y)(\frac{\pi}{2}-x)=\pi(\frac{\pi}{2}-x)$, so $f(x,y)$ should contain the terms $-xy$, afterthat we have $f(x,y)=g(x,y)-xy$ and $g(\frac{\pi}{2}-y,\frac{\pi}{2}-x)=g(y+\frac{\pi}{2},x-\frac{\pi}{2})$. Consider the period related to π, we consider the trigonometric function while $f(x,y)=2(\sin x+\cos y)-xy$ is the answer.

Example 6.4.21 *Given $f(n,z)=f(n-1,x)+n+x$ for n is a positive integer and $f(0,x)=x$. Find the general term of $f(n,x)$.*

We have $f(n,x)=(n+x)+(n-1+x)+...+(1+x)+x=(1+2+...n)+(n+1)x=\frac{(n+2x)(n+1)}{2}$.

Exercise:

1. Rewrite $(2n-1)(2n+1)(2n+3)$ in forms of $A+B(2n)+C(2n)(2n-1)+D(2n)(2n-1)(2n-2)$, where A,B,C and D are constants independent of n.
2. Factorize $2x^2-3xy-2y^2+2x+11y-12$.
3. Factorize $6x^2+xy-y^2-3x+y$.
4. Factorize $\sum\limits_{cyc} f$ where $f(a,b,c)=(a+b)^3(a-b)$.
5. Factorize $\sum\limits_{cyc} f$ where $f(a,b,c)=a^2b^2(a-b)$.
6. Factorize $\sum\limits_{cyc} f$ where $f(a,b,c)=(a-b)^5$.
7. Factorize $\sigma_2^3-\sum\limits_{cyc} f$ where $f(a,b,c)=a^3b^3$.

8. Show that for $\sum\limits_{cyc} f$ where $f(a,b,c) = a^n(b -^s -c^s)$ (n and s is positive integer), then $\prod\limits_{cyc}(x-y)$ is always a factor of it.
9. Show that all elementary functions are symmetrical.
10. Factorize $1 + \sin 2x$.
11. Show that for a polynomial with two variables, then it is cyclic if and only if it is symmetrical.
12. Show that for a polynomial with three variables, if it is cyclic, then it is either symmetrical or alternating.
13. A real function $f(x,y)$ is cyclic and obey $f(x, y-1) = f(x,y) - y + 1$. Find $f(x,y)$.
14. Given that $f(c,x) = \frac{cf(c-1,x)}{c+1}$ where c is a positive integer, find the general term of $f(c,x)$.

6.5 Functional Equation

Definition 6.5.1 *Functional equation solves a function under some given condition.*

We have already encountered some of them in the previous section and we will mainly introduce the skill of substitution to solve function here.

Definition 6.5.2 $f_n(x) = f(f_{n-1}(x)) = ... = f(f(...f(x)...))$ *where function is operated n times.*

Example 6.5.3 *If* $f(x) = x^2 - 1$, *then* $f_2(x) = (x^2-1)^2 - 1 = x^4 - 2x^2$.

Lemma 6.5.4 *If* f *is a polynomial and* $\deg f = k$, *then* $\deg f_n = k^n$.

The proof is quite obvious: Let $f(x) = a_k x^k + r(x)$ where $r(x)$ is the remaining terms of $f(x)$ while $\deg r < k$. Define $g(x) = f(x) - r(x)$. Then $g_n(x) = (g(x))^n$, and therefore $\deg f_n \deg g_n = (\deg g)^n = k^n$.

An interesting corollary can be deduced.

Corollary 6.5.5 *Given* $f(x)$ *is a linear function. If* x *is a root of* $f(x) = x$, *then it's also the root of* $f_n(x) = x$.

Proof: Let $f(x) = ax+b$. Root of $f(x) = x$ is $x = \frac{b}{1-a}$. $f_n(x) = a(a(...(ax+b)+b+...)) + b = a^n x + a^{n-1}b + a^{n-2}b + ... + ab + b = x$. Arranging term gives $x = \frac{a^{n-1}b + a^{n-2}b + ... + ab + b}{1-a^n} = \frac{b}{1-a}$.

Now we may given some function that $f(x)$ is not directly given and we have to find it out.

Example 6.5.6 *If $f(2x-1) = 8x-4$, find $f(5)$.*

Method 1: Let $u = 2x-1$. Then $f(u) = 4u$ and $f(5) = 20$. Another method is to let $2x-1 = 5$, then $x = 3$ and $f(2x-1) = 8x-4 = 20$.

Example 6.5.7 *If $f(x^2-1) = x^4-1$, find $f(10)$.*

Let $u = x^2-1$. Then $f(u) = (x^2-1)(x^2+1) = u(u+2)$, $f(10) = 120$.

Example 6.5.8 *If $f(m)$ is a function which domain and co-domain is natural number, satisfies $f(m+n) = f(m)+f(n)+mn$ and $f(1) = 1$. Find $f(x)$.*

Observe that $f(n) = f(n-1)+f(1)+(n-1) = f(n-1)+n = f(n-2)+(n-1)+n = ... = 1+2+...+n = \frac{n(n+1)}{2}$. Note that in fact, this rule given is applicable over real number.However, it's proof can be much complicated. Moreover, functional equation can be very hard. Now we can take a look at a harder problem about function.

Example 6.5.9 *Show that no $f(m)$ exist which domain and co-domain is natrual number and satisfy $f(m+n) = f(m+f(n))+mn$.*

Consider $f(1+1) = f(1+f(1))+1 = .. = f(1+f_n(1))+\sum f_i(1) \geq n$. Therefore for any n the corresponding value is unknown. Therefore no such functino exists.

Example 6.5.10 *Show that no real function f exists that for all real x, y, z, $f(x+y+z) = \sum\limits_{cyc} f(x)+\sigma_3$.*

Firstly we have $f(1+0+0) = f(1)+2f(0)+0$ and therefore 0 is a fixed point of f. Now $f(2) = 2f(1)+0$ and $f(3) = 3f(1)+1 = f(2)+f(1)+0$ which makes a contradiction. Therefore no such function exists.

Example 6.5.11 *φ and λ are two non-constant functions satisfying $\varphi(x+t)+\varphi(x-t)-2\varphi(x) = \lambda(t)$. Show that φ can't have both absolute maximum and minimum. (Q.6a, AL 1981)*

Fix t. We will find that LHS is fixed no matter how x changes. Therefore let two real number $x_1 < x_2$.

We will have $\lambda(t) = (\varphi(x_1+t)-\varphi(x_1))-(\varphi(x_1)-\varphi(x_1-t)) = (\varphi(x_2+t)-\varphi(x_2))-(\varphi(x_2)-\varphi(x_2-t))$.

Therefore we know that the difference of rate of change of φ is changing in a constant rate. Since λ is not identically zero φ must diverge.

Note: this solution is not the official ones. The official solution proves by contradiction: let the absolute maximum and minimum of the function exist, and find that $0 \leq \varphi \leq 0$.Now look at the next part of the question:

Example 6.5.12 *Given φ is differentiable, show that $\varphi'(x+y)-\varphi'(x) = \varphi'(y)-\varphi'(0)$.*

Since φ is differentiable, λ must be differentiable too. Now differentiate the given equation with respect to t and eliminate the $\varphi(x)$, we have $\varphi;(x+t)+\varphi;(x-t)=\lambda'(t)$. Substitude $x+t$ into x we have $\varphi'(x+2t)+\varphi'(x)=\lambda'(t)$. By setting $t=\frac{y}{2}$ we have $\varphi'(x+y)-\varphi'(x)=\lambda'(\frac{y}{2})=\varphi'(y)-\varphi'(0)$.

Example 6.5.13 *Show that $\varphi''(x)$ is identically equal to $\varphi''(0)$, hence show that it is a polynomial with degree lower than 2.*

We have $\frac{\varphi'(x+y)-\varphi'(x)}{y}=\frac{\varphi'(y)-\varphi'(0)}{y}$. By taking limit to zero we get $\varphi''(x)$ is identically equal to $\varphi''(0)$. The last part of the question is obvious.

Exercise:

1. Let $f(x)=3x-7$. Solve $f_{29}(x)=x$.
2. f is a linear function that $f(3x-1)=x+9$, find $f(x)$.
3. f is a polynomial that $f(x^2+3)=x^2-2x+4$, find $f(x)$.
4. f is a function that $f(x+x^{-1})=x^3+3+x^{-3}$, find $f(x)$.
5. If $f_n(x)=2f_{n-1}(x)-1$, find $f(x)$.
6. If $f_n(x)-f_{n-1}(x)=3xf_{n-1}(x)$, show that $f((3x+1)^n)=(3x+1)^{n+1}$.

6.6 Partial fraction

We rewrite a fraction into a partial fraction by the following rules.

1. Factorize the denominator.
2. Consider a fraction $\frac{f(x)}{\prod(g_i(x))^{k_i}}$, where every f,g are polynomial and every $g_n(x)$ is irreducible over integer coefficient.
3. It can be rewrite in the forms of $\sum\frac{c_i}{g_i}$.
4. If $k_i=r>1$ for some g, then $\sum\limits_{i=1}^{r}\frac{c_i}{(g_i)^i}$ have to be added to the sum.
5. $f(x)$ satisfies $uf(u)=f(u-x)f(x)$ for all non-zero u and real x. At the same time 0 is its fixed point. Show that $f(x)$ is identically zero.

We can determine whether the polynomial is irreducible by the following method.

Theorem 6.6.1 *The remainder theorem states that when a polynomial $f(x)$ is divided by $(x-a)$ while the remainder is r, then $f(a)=r$.*

Proof: By Euclidean logarithm, we can know that in division of polynomial there's a unique pair of $f(x)=q(x)(x-a)+r$. Then obviously $f(a)=(a-a)q(a)+r=r$.

Theorem 6.6.2 *The factor theorem states that a is a root of $f(x)$ is equivalent to $f(x)$ is divisible by $(x-a)$.*

The proof is left for readers.

Theorem 6.6.3 *Eisenstein's irreducibility criterion is a sufficient condition to show that a polynomial is unable to factorize to factor with rational coefficients. i.e, it won't have rational root. It won't have rational factor if there's a prime p which divides all terms except the leading term, while p^2 does not divide the constant term.*

Note that it's only a sufficient criteria. Even it does not stand, it don't implies that the polynomial is reducible over rational factor.

Example 6.6.4 x^2+3x+6 *is irreducible since 3 divides all terms except the leading terms, 9 does not divide the constand term.*

Example 6.6.5 x^2+1 *is irreducible. However we can't find aw prime to satisfy the criteria. Now we substitute $x=u+1$ to get u^2+2u+2 which is clearly irreducible by considering $p=2$.*

One of the most famous partial fraction is $\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$.

Example 6.6.6 *Evaluate* $\frac{1}{1\times 2}+\frac{1}{2\times 3}+...+\frac{1}{2010\times 2011}$.

Clearly it's equal to $1-\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2009}-\frac{1}{2010}=\frac{2009}{2010}$.

Example 6.6.7 *Rewrite* $(3n+1)(3n+2)$ *rewrite in forms of* $A+B(n+1)+C(n+1)(n+2)$.

Expanding LHS gives $9n^2+9n+2$. Expanding RHS gives $Cn^2+(3C+B)n+(2C+B+A)$. By comparing the coefficient, we have $C=9$, $B=-18$ and $A=2$. Therefore $(3n+1)(3n+2)=9(n+1)(n+2)-18(n+1)+2$.

Example 6.6.8 *Rewrite* $\frac{2}{x^2-1}$ *to partial fraction.*

Since $x^2-1=(x-1)(x+1)$, we let $\frac{2}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}=\frac{(A+B)x+(A-B)}{x^2-1}$. So we have the simutaneous equation $\left\{\begin{array}{c}A+B=0\\A-B=2\end{array}\right.$, so that $A=1$ and $B=-1$. So $\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}$.

Example 6.6.9 *Rewrite* $\frac{2x+1}{x(x-1)^2}$ *into partial fraction.*

Consider $\frac{2x+1}{x(x-1)^2}\equiv\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{(x-1)^2}=\frac{Cx^3+(A+B+C+C)x^2+(-2A-B+D)x+A}{x(x-1)^2}$. By comparing the coefficiet, we have $C=0$, $A=1$ and $\left\{\begin{array}{c}A+B+C+D=0\\-2A-B+D=2\end{array}\right.$.

Solving the equation gives $B = \frac{-5}{2}$ and $D = \frac{3}{2}$. Therefore $\frac{2x+1}{x(x-1)^2} = \frac{1}{x} - \frac{5}{2(x-1)} + \frac{3}{2(x-1)^2}$.

From the above example, you can see that calculation involving partial fraction requires the abilities of solving simutaneous equation for 3 or even more variables. Now we consider an A-level question:

Example 6.6.10 *Resolve* $\frac{7x+9}{x(x+1)(x+3)}$ *into partial fraction; express* $\sum_{k=1}^{n} \frac{7x+9}{x(x+1)(x+3)}$ *in forms of* $A + \frac{B}{n+1} + \frac{C}{n+2} + \frac{D}{n+3}$*, where A,B,C,D are constants, and hence evaluate its infinity sum. (AL 2008)*

The first part is left for readers, where the answer is $\frac{3}{x} - \frac{1}{x+1} - \frac{2}{x+3}$. So the sum can be expressed in the form $\sum[(\frac{1}{x} - \frac{1}{x-1}) + 2(\frac{1}{x} - \frac{1}{x+3})]$, while terms eliminating each other, we have $(1 - \frac{1}{n+1}) + 2(\frac{11}{6} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}) = \frac{8}{3} - \frac{3}{n+1} - \frac{2}{n+2} - \frac{2}{n+3}$. In the infinity sum, n goes to infinity so the last three terms is neglected, and the infinity sum is $\frac{8}{3}$.

Note: consider a fraction $\frac{f}{g}$ where $\deg f > \deg g$, and $f(x) = g(x)q(x) + r(x)$ (Euclidean alogarithm), then $\frac{f}{g} = q(x) + \frac{r(x)}{g(x)}$ and we express the partial fraction on $\frac{r(x)}{g(x)}$ only.

Exercise:

1. Express $\frac{x+1}{x^2+5x+6}$ in terms of partial fractions.
2. Express $\frac{3x}{(x-4)(x+2)}$ in terms of partial fractions.
3. Express $\frac{x^3}{16-x^2}$ in terms of partial fractions.
4. Express $\frac{1-x}{1+x+x^2+x^3}$ in terms of partial fractions.
5. Express $\frac{81}{x(9+x^2)^2}$ in terms of partial fractions.
6. Express $\frac{x^3(x^2+4)}{(x^2+2)^3}$ in terms of partial fractions.
7. Show that following the rule, the partial fraction is unique.

Chapter 7

Sequence

7.1 Nature of sequence

We can treat a sequence as a discontinue version of a function. For example consider $a_1 = 1$ and $a_{n+1} = 1+a_n$, we can that $a_n = n$. Also it can be a function with domain and co-domain of natrual integer (or non-negative function), where example 6.5.8 is an example.

Sequence can be considered a set of solution when one of the variables varies.

Example 7.1.1 *For the equation $xy = 1$, $y_x = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ where y_x is the solution of y for different x, and $y_x = x^{-1}$.*

7.2 Recurrence sequence

Definition 7.2.1 *Recurrence sequence is something that the term can be expressed by a function of previous term.*

The degree of the sequence is determined by number of variables required by the function. For example, $a_n = f(a_{n-1})$ is a recurrence sequence of the first degree.

A quick note, the first degree recurrence sequence sometimes can be simplified into other types sequence.

Lemma 7.2.2 *$a_n = f(a_{n-1}) = a_{n-1} + c$ is the arithmetic sequence.*

Lemma 7.2.3 *$a_n = f(a_{n-1}) = ca_{n-1}$ is the geometric sequence.*

Sometimes it can be composed function.

Example 7.2.4 *$a_n = 2a_{n-1} - 3$, $a_1 = 4$, express $f(n)$ such that $f(n) = a_n$ for positive integers.*

Observe that $2(3+1)-3 = 3+2 = 5$, $2(3+2)-3 = 3+4 = 7$. $2(3+4)-3 = 3+4+4 = 11$, so we can find that $a_n = 3 + 2^{n-1}$ by mathematical induction.

Example 7.2.5 *Given $a_1 = 1$, $(n-1)a_n = 2\sum\limits_{i<n} a_i$. Find its general term.*

We can show that it's monotonically increasing. i.e., $a_i > a_j$ for all $i > j$. Denote $a_n = k$ and $S_n = a_1 + a_2 + ... + a_n$. Then $a_n = \frac{2S_{k-1}}{n-1} = \frac{2S_k - 2k}{n-1}$ and $a_{n+1} = \frac{2S_k}{n}$. By subtracting, $a_{n+1} > a_n$ if and only if $\frac{2k}{n-1} > \frac{2S_k}{n(n-1)}$, and $k > \frac{S_k}{n}$ implies it's monotonically increasing.

Subtracting $na_{n+1} = 2S_k$ by $(n-1)a_n = 2S_{k-1}$, we have $a_{n+1} = \frac{(n+1)a_n}{n}$ which gives $a_n = n$.

Example 7.2.6 *Tower of Hanoi. Denote a_n as the minimum steps needed for n rings.*

Let $a_n = k$. For $n+1$ rings, we have to move the first n rings to a coloumn to another, move the largest one, then recover the n rings to the location of largest ring. Therefore $a_{n+1} = 2a_n + 1$. Consider $f(n) = 2n + 1$, then we have $a_n = f_{n-1}(a_1) = 2^k + 2^{k-1} + ... + 2 + 1 = 2^k - 1$. Therefore $a_n = 2^k - 1$.

Example 7.2.7 *Find the general term of sequence $a_n = xa_{n-1} + y$*

By subtracting $a_n - a_{n-1} = x(a_{n-1} - a_{n-2})$, we know that $b_n = a_n - a_{n-1}$ is a geometric sequence. Then $b_n = x^{n-2}(a_2 - a_1$. Then the sum $a_n = a_1 + b_1 + b_2 + ... + b_{n-1} = a_1 + (1 + x + x^2 + ... + x^{n-2})(a_2 - a_1) = a_1 + (a_2 - a_1)\frac{(1-x^{n-1}}{1-x}$.

Example 7.2.8 *Given a sequence satisfying $S_n = 4a_n + 3n - 4$ where S_n is the sum of first n terms. Find its general term and the general term of sum.*

By rearranging the terms, we have $a_n = \frac{s_{n-1}+4}{3} - n$. By the difference $a_n - an - 1$ we obtain $a_n = \frac{4a_{n-1}}{3} - 1$. Consider $a_1 = S_1 = \frac{1}{3}$ and $a_2 = \frac{-5}{9}$, $a_n = \frac{1}{3} + (\frac{-5}{9} - \frac{1}{3})(1 - (\frac{4}{3})^{n-1})(1 - \frac{4}{3})^{-1} = 3 - \frac{8(4^{n-1})}{3^n}$.

By the sum $S_n = 4a_n + 3n - 4$, we have $S_n = 4(3 - \frac{8(4^{n-1})}{3^n}) + 3n - 4 = 8 + 3n - 8(\frac{4}{3})^n$.

7.3 Characteristic equation

Theorem 7.3.1 *A Characteristic equation $x^2 - px - q = 0$ can be formed from the recurrence sequence $a_{n+2} = pa_{n-1} + qa_{n-2}$, then $a_n = k_1x_1^n + k_2x_2^n$ where k are coefficients and x are the solution of the characteristic equation.*

Proof is split into different cases, the core idea gives the main apporach to put the sequence in forms of geometric sequence, while the special cases consider it's determinant ($\Delta = p^2 - 4q$) or sum of p and q. If $p + q = 1$, then the case can also be simplified.

Core idea

Select x_1 and x_2 such that $x_1 + x_2 = p$, $x_1x_2 = -q$. Then $a_n = (x_1 + x_2)a_{n-1} - x_1x_2a_{n-1}$. Rearranging gives $a_n - x_1a_n = x_2(a_{n-1} - x_1a_{n-2})$ which gives a geometric sequence, therefore $a_n - x_1a_n = x_2^{n-2}(a_2 - x_1a_1)$. Similarly, $a_n - x_2a_n = x_1^{n-2}(a_2 - x_2a_1)$.

Then we have $\begin{cases} a_{n+1} = x_1a_n + x_2^n(a_1 - x_1a_0) \\ a_{n+1} = x_2a_n + x_1^n(a_1 - x_2a_0) \end{cases}$ By considering the difference we get $a_n = k_1x_1^n + k_2x_2^n$ where $k_1 = \frac{a_1 - a_2x_0}{x_1 - x_2}$ and $k_2 = \frac{x_1a_0 - a_1}{x_1 - x_2}$.

Special case: $\Delta = 0$

Under this special condition, the coefficient can be simplified. In this case $x_1 = x_2$.

By similar arguement, starting from $a_n = x_1a_{n-1} + x_2^{n-1}(a_1 - x_1a_0) = x_1a_{n-1} + x_1^{n-1}(a_1 - x_1a_0)$, we have

$$x_1a_{n-1} = x_1^2a_{n-2} + x_1^{n-1}(a_1 - x_1a_0)$$

$$x_1^2a_{n-2} = x_1^3a_{n-3} + x_1^{n-1}(a_1 - x_1a_0)$$

Summing up gives $a_n = x_1^na_0 + nx_1^{n-1}$.

Special case: $\Delta < 0$

When the roots of the equation is complex number, we can put it in forms of $rcis\theta$. If one of the root is $rcis\theta$, then another root will be $rcis(-\theta)$.

$k_1x_1^n + k_2x_2^n = k_1r^n(cis(n\theta)) + k_2r^n(cis(-n\theta)) = r^n[(k_1 + k_2)\cos n\theta + (k_1 - k_2)\sin n\theta]$.

Special case: $p + q = 1$

Here, $a_{n+1} - a_n = -q(a_n - a_{n-1})$. Then $a_n = (a_2 - a_1)(-q)^{n-2} + a_{n-1} = (a_2 - a_1)((-q)^{n-2} + (-q^{n-3})) + a_{n-2} = ... = a_1 + (a_2 - a_1)(1 - q + q^2 - ... + (-q)^{n-2}) = a_1 + \frac{(a_2 - a_1)(1-(-q)^{n-1})}{1-q}$.

Theorem 7.3.2 *Given a recurrence sequence $a_n = pa_{n-1} + qa_{n-2}$, consider $x^2 - px - q = 0$, if the roots are x_1 and x_2, then its general term will be $k_1x_1^n + k_2x_2^n$. The unknown coefficient can be solved through checking the first few terms.*

Example 7.3.3 *The Fibonacci sequence: $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ for all $a \geq 3$. Find its general term.*

By Solving $x^2 - x - 1 = 0$, we have $a_n = k_1(\frac{1+\sqrt{5}}{2})^n + k_2(\frac{1-\sqrt{5}}{2})^n$. By considering the first three terms, we have $k_1 = k_2 = \frac{1}{\sqrt{5}}$. Therefore $a_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n]$.

Theorem 7.3.4 *While considering characteristic equation, constant term can be eliminated. i.e., for $a_n = pa_{n-1} + qa_{n-2} + r$, then the characteristic equation is still $x^2 - px - q = 0$, however the general term becomes $a_n = k_1x_1^n + k_2x_2^n + k_3$, the x_i are roots of the characteristic equation, while the k_i are coefficients.*

The proof is left for readers, provided the main idea is that when we are comparing difference, the constant terms can be eliminated.

Example 7.3.5 *Given* $a_n = 2a_{n-1} + 3a_{n-1} - 4$*,* $a_1 = 1$*,* $a_2 = 2$*. Find its general term.*

By the characteristic equation $x^2 - 2x - 3 = 0$, we have $x_1 = -1$ and $x_2 = 3$. Then $a_n = k_1 3^n + k_2(-1)^n + k_3$. By considering $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$, we have $a_n = \frac{1}{4}(3^{n-1} + (-1)^n) + 1$.

Example 7.3.6 *Given* $a_n = 2a_{n-1} - a_{n-2}$ *for all* $n \geq 3$*.* $a_1 = 3$*,* $a_2 = 4$*. Find the genreal term.*

Method 1: Since $p + q = 1$ in RHS, apply $a_n = a_{n-1} + (a_2 - a_1)(-q)^{n-2} = a_{n-1} + (4-3)(-(-1))^{n-2} = a_{n-1} + 1$. Therefore $a_n = n + c$ where c is a constants. By comparing a_1, $a_n = n + 2$.

Method 2: Solving the characteristic equation $x^2 - 2x + 1 = 0$ gives $x_1 = x_2 = 1$, then $a_n = k_1(1)^n + k_2$, by comparing a_1 and a_2, we have $a_n = n + 2$.

Now we can try to extend the idea of characteristic equations for recurrence sequence that not in the senond degree. For the first degree, if $a_n = pa_{n-1}$, then it's a geometric sequence, so it's general term will be $k_1 x^n$. The x is strictly p. But you can also think in another way. Consider its characteristic equation $x = p$, we also obtain the common ratio of the geometric sequences.

A more common techinique is that when it is given $a_n = pa_{n-1} + qa_{n-2}$, we can think that $x^n = px^{n-1} + qx^{n-2}$. We will obtain many zeros and two non-trivial solution from the equation, and the general term can be constructed by the powers of the root. Of course, the powers of zeros can be eliminated.

Therefore when we consider the recurrence in the third degree or even more, we can also find the characteristic equation, solve them and find the general term.

Example 7.3.7 *A recurrence sequence* $a_n = a_{n-1} + ka_{n-2}$ *satisfies* $a_i + a_{i+1} + a_{i+2} + ... + a_{i+9} = 11a_{i+6}$*. Find* k*.*

In order to solve this one, the idea is rather simple, but the calculation is complicated.

Firsyly we express a_{i+n} in terms of a_i and a_{i+1}. i.e., $a_{i+n} = p_n a_i + q_n a_{i+1}$, where p and q are polynomials with unknown k. Firstly we show that $kq_n = p_{n+1}$ and $p_n + q_n = q_{n+1}$ by mathematical induction. considering $p_n = \{1, 0, k, k, ...\}$ and $q_n = \{0, 1, 1, k+1, ...\}$, it's true when $n = 1$. Let the conjecture be ture for $n = k$. Then $p_{n+1} + kp_n = kq_n + kp_n = kq_{n+1} = p_{n+2}$ and $q_{n+1} + kq_n = q_{n+1} + p_{n+1} = q_{n+2}$.

Then we can construct the sequence:

$$p_n = \big\{1, 0, k, k, k^2 + k, 2k^2 + k, k^3 + 3k^2 + k, 3k^3 + 4k^2 + k, ...\big\}$$
$$\big\{...k^4 + 6k^3 + 5k^2 + k, 4k^4 + 10k^3 + 6k^2 + k, ...\big\}$$
$$q_n = \big\{0, 1, 1, k+1, 2k+1, k^2 + 3k + 1, 3k^2 + 4k + 1, k^3 + 6k^2 + 5k + 1, ...\big\}$$
$$\big\{...4k^3 + 10k^2 + 6k + 1, k^4 + 10k^3 + 15k^2 + 7k + 1...\big\}$$

Sum the first 10 terms up and compare with p_{i+6} and q_{i+6}, we have two equations but one unknown only. Therefore we can consider one of them: $5k^4 + 20k^3 + 21k^2 + 8k + 1 = 11(k^3 + 3k^2 + k)$, by factor theorem and Eiseistein's theorem, we can find that $k = 1$ is the only real and integer solution. Checking another equation: $k^4 + 15k^3 + 35k^2 + 28k + 9 = 11(3k^2 + 4k + 1)$ gives the same solution. Therefore $k = 1$.

Exercise:

1. Express the general term of a_n, given $a_n = a_{n-1} + c$, $a_1 = k$, in terms of recurrence sequence.
2. When $a_n = pa_{n-1} + qa_{n-2} + r$, show that the characteristic equation is valid.
3. Given $a_n = x^2 a_{n-2}$, express the general term in terms of a_1.
4. Given $a_n = a_{n-1} - a_{n-2}$, $a_1 = 1$, $a_2 = 0$, show that $a_n = \frac{(-1)^n}{\sqrt{3}i}(w^n - w^{2n})$, where $w = \frac{1+\sqrt{3}i}{2}$.
5. Given $a_n - a_{n-1} = n - \frac{1}{n(n+1)}$ and $a_1 = 2$, find its general term.
6. Given $a_{n+1} = 2S_n + n^2 - n + 1$ and $a_1 = 1$, find its general terms.
7. Given $a_1 = 10$, $a_{n+1} = (a_n)^{\frac{1}{n}}$, find its general term.
8. Given $a_{n+3} = 3a_{n+2} + 3a_{n+1} + a_n$ and $a_1 = 3$, find its general term.
9. Given $a_{n+2} = a_{n+1} + 2a_n$. Consider $a_n = p_n a_1 + q_n a_2$, show that $|p_n - q_n| = 1$, moreover $p_n - q_n = (-1)^{n+1}$.

7.4 Finite difference

Example 7.4.1 *Given a sequence $a_n = P(n)$ where P is a polynomial, If $a_n = \{0, 3, 8, 15, ...\}$, find its general term.*

Now we construct a sequence $b_n = a_{n+1} - a_n$, then $b_n = \{3, 5, 7, ...\}$. Obviously $b_n = 2n + 1$. Then we can get $a_n = a_1 + b_1 + b_2 + ... + b_{n-1} = 2(1 + 2 + ... + n - 1) + (n - 1)(1) = n^2 - 1$.

Such a method is called finite difference. Considering the difference of terms, we try to construct the general term (usually polynomial) of the sequence. We starts from the analysis of polynomial.

Theorem 7.4.2 *Given $\deg f = k$ and coefficient of every term in different degree are not zero, then deg$(f(x+1) - f(x-1)) = k - 1$.*

The main idea of this statement is to show that while considering the difference of the same polynomial, only the term of highest degree is sure to be eliminated.

Proof: We now start from a easier statement: Let $f(x) = x^n$. Then $f(x+1) - f(x-1) = 2nx^{n-1} + R(x)$ where R is other terms of the difference. It can be easily deduced by binomial theorem.

Now consider $f(x) = \sum^n a_i x^i$ where $a_i \neq 0$. Then $f(x+1) - f(x-1) = \sum a_i((x+1)^i - (x-1)^i)$. By the previous claim, we can show that the coefficient of x^{n-1} isn't zero unless a_n is zero. But then this will violate the assumption $\deg f = k$. That finishes the claim.

The case considering difference of $f(x) - f(x-1)$ whould be more obvious for the above claim. Therefore we can go to the main concepr of the finite difference:

Example 7.4.3 *Again find the general term of* $a_n = \{0, 3, 8, 15, 24, 35, ...\}$

The first time difference gives $\{3, 5, 7, 9, 11, ...\}$ and the second time difference gives $\{2, 2, 2, 2, ...\}$. Since the difference of the sequence is a constant, then it must be a quadratic polynomial. By solving equation we get $a_n = n^2 - 1$.

Core idea

If the n^{th} difference is a constant, then $\deg f = n$. It is because when $f = \sum a_i x^i$, then $f^{(n)} = a_n n!$ which is a constant. If variables exists in the sequence, then it can not be a constant (because $f(x) = c$ can not have infinite roots) or else it violates our assumption of degree. Moreover, the finite difference is valid because in $f^{(n)}$, every point on the graph gives the same slope, therefore the discrete version (the finite difference) gives constant as well. We can obtain the coefficent of x^n by the following method:

Corollary 7.4.4 *Let the constant of n^{th} difference be k. Then the coefficient of x^n is $\frac{k}{n!}$.*

It can be easily deduced from the above explaination.

Example 7.4.5 *Find the general term of* $a_n = \{-4, 4, 24, 62, 124, 216, ...\}$

The first difference is $\{8, 20, 38, 62, 92, ...\}$, the second difference will be $\{12, 18, 24, 30, ...\}$ while the thirs difference is $\{6, 6, 6, ...\}$. Therefore it contains a term x^3. Now considering $b_n = a_n - n^3 = \{-5, -4, -3, -2, -1, 0, ...\}$, we can get $b_n = n - 6$ so that $a_n = n^3 + n - 6$.

Now we try to find the characteristic of finite difference on other functions.

Theorem 7.4.6 *The finite difference of an exponential series tends to be similar in common ratio.*

Proof: Let $f(x) = c^x$ and a_n is its discrete version of f. Then $a_n - a_{n-1} = (c-1)a_{n-1} = c^{n-1}(c-1)a_1$. By considering the difference of $b_n = a_n - a_{n-1}$ again we will get the same common ratio c. An induction will finish the calim.

When we get the finite difference, setting a recurrence relationship will gives the answer.

Another approach is that $(c^k)' = \ln c c^k$ so that it is similar with same common ratio.

Now we denote $a_n^{(k)}$ is the n^{th} term of the k^{th} difference of the sequence.

Theorem 7.4.7 *When $a_n = c^n$, then $a_n^{(k)} = c^n(c-1)^k$.*

The proof is left for readers.

Example 7.4.8 *Find the general term of* $\{1, 3, 7, 15, 31, 63, ...\}$

The first and senond finite difference is the same: $\{2, 4, 8, 16, 32, ...\}$ Then we have $c - 1 = 1$ since the two difference is identical. i.e., the second difference is multiplied by one. Therefore it contains the term 2^n. By considering the reaminders, we will have $a_n = 2^n - 1$.

When the sequence is composed by composed function, the case will be more complicated.

Example 7.4.9 *Find the general term of* $\{3, 8, 17, 32, 57, 100, 177, ...\}$

We know that if it is polynomial, it will be eliminated after finite times of difference.

The first three difference are $\{5, 9, 15, 25, 43, 77, ...\}$, $\{4, 6, 10, 18, 34, ...\}$ and $\{2, 4, 8, 16, ...\}$ respectively. Then it contains the term 2^n. Eliminating the exponontial term gives $a_n = n^2 + 2^n$.

It is hard to find the general term of sequence composed by two exponential function if some terms are given only, but we acn finish it by approximation tricks.

Let $a_n = x^n + y^n$. When n is odd, it equals to $(x + y)(x^{n-1} - x^{n-2}y + ... + y^{n-1})$. When n is even, it approximately to. So for odd terms, it is divisible by $(x + y)$. You may also treat this as a signal of two exponential function.

Example 7.4.10 *Find the general term of* $\{5, 13, 35, 97, 275, 793, ...\}$.

Observe that the odd terms are multiples of 5, so it should be composed by power of 2 and 3. Solving gives $a_n = 2^n + 3^n$.

It can also help us to identify the recurrence relationship.

Example 7.4.11 *(Fibonacci sequence)Find the recurrence relationship of the sequence* $\{1, 1, 2, 3, 5, 8, ...\}$.

We have the first time difference as $\{0, 1, 1, 2, 3, 5, 8, ...\}$. When we arrange the difference and the original sequence together, we will find that $a_n = a_{n+2}^{(1)} = a_{n+2} - a_{n+1}$ which gives $a_{n+2} = a_{n+1} + a_n$.

One important note: pure exponential series will not contain any negative value! If it gives some negative value, or not icreasing/decreasing monotonically, it must not be pure exponential series.

Here is a quick remainder on analysis on finite difference about finishing the idea inversely.

Theorem 7.4.12 *If f is a polynomial while a_n is the discrete version of f, then $a_n^{(1)} = \frac{1}{2}(f'(x+1) - f'(x))$.*

The proof is left for readers.

Example 7.4.13 *Let $f(x) = x^3+x-6$, then $\frac{1}{2}(f'(x+1)-f'(x)) = 3x^2+3x+2$ which suits the finite difference $\{8, 20, 38, ...\}$.*

Exercise:

1. Given $\deg f = k$ and coefficients in each terms are non-zero reals, show that $\deg(f(x+c) - f(x-c)) = k-1$.
2. Evaluate $a_n^{(1)} - a_n^{(2)}$.
3. Prove theorem 7.4.7.
4. Prove, in terms of finite difference, that exponential functino is increasing faster than polynomial generally.
5. Show that if $a_n = b_n + c_n$, then $a_n^{(k)} = b_n^{(k)} + c_n^{(k)}$.
6. Find the general term of $\{-2, -1, 8, 31, 74, 143, 244, ...\}$
7. Find the general term of a_n if a_1, $a_1^{(1)}$, $a_1^{(2)}$, $a_1^{(3)}$ and $a_1^{(4)}$ are 1,15,50,60 and 24 respectively.
8. Express $a_n^{(k)}$ in terms of a_i. Hence, show that when a_n is a polynomial of degree k, $a_n^{(q)} \equiv 0$ when $q > k$.
9. Find the recurrence relationship of $\{1, 1, 0, -1, -1, 0, 1, 1, ...\}$.
10. find the general term of $\{-4, 2, 20, 74, 236, ...\}$.
11. Find the general term of $\{8, 26, 92, 338, 1268, 4826, ...\}$. (Difficult)
12. Find the general term of $\{13, 85, 559, 3697, 24583, ...\}$.
13. Find the general term of $\{-28, -16, 44, 266, 1088, 4244, 16652, ...\}$.
14. Given $a_n = \{1, -1, 1, -1, ...\}$. Show that $a_n^{(k)} = 2^k(-1)^{k+n}$.
15. Prove theorem 7.4.12.
16. Can you think of any method to find the sequence composed by three exponential function systematically? How about trigonometry function, integer function or transendental function?
17. Will theorem 7.4.12 be valid if f isn't a polynomial? Why?
18. Show that for $a_n = \left[\frac{n}{p}\right]$, then $a_n^{(k)}$ is periodic for every p for all k. Hence show that if $a_n^{(k)}$ is periodic, then $a_n^{(r)}$ is periodic for all $r \geq k$ with the same period.

7.5 Convergence

Definition 7.5.1 *The indeterminate form is the algebraic expressions that we can not determine its value.*

The indeterminate form usually comes from limit, including $\frac{0}{x}$, $\frac{x}{0}$, $\frac{\infty}{\infty}$, 1^∞, etc. Note that ∞ or $\frac{0}{x}$ is not indeterminate form.

We have to use other methods to obtain the limit.

Theorem 7.5.2 *(L'Hopital's rule) If* $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = 0$ *or* $\lim_{x\to p} f(x) = \pm \lim_{x\to p} g(x) = \pm\infty$*, and* $\lim_{x\to p} \frac{f'(x)}{g'(x)} = L$*, then* $\lim_{x\to p} \frac{f(x)}{g(x)} = L$*.*

The proof is omitted.

Example 7.5.3 *Evaluate* $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Of course you can simply reduce the common factor, but we can also apply the rule. Since both will be zero when the we limit x towards 1, so we apply the rule once to get $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 2$.

Now we further simplify our expressions in sequences. If $a_n = f(n)$ for every non-negative (or natrual number, depends whether a_0 exists) integer n, then we write $\{f(n)\}$.

When we are finding the value of $\lim_{x\to\infty} \frac{F(x)}{G(x)}$, we are actually finding the rate of increasing of the two functions. We will talk about this idea later in this section.

Example 7.5.4 *Evaluate* $\lim_{x\to\infty} \frac{\ln x}{x}$.

We see both denominator and numerator tends to infinity as x goes to infinity, so we apply the rule.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{x^{-1}}{1} = 0$$

The following limit is one of the most famous indeterminate form:

Example 7.5.5 *Evaluate* $\lim_{x\to 0} \frac{\sin x}{x}$.

In fact, it can be done by geometric method but we try to finish it by algebraic method here. Since it becomes $\frac{0}{0}$, so we apply the rule once.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Example 7.5.6 *Evaluate* $\lim_{x\to 0} \frac{\sin x^2}{x^2}$.

Substitude $u = x^2$, then it becomes $\lim\limits_{u \to 0} \frac{\sin u}{u} = 1$.

Finding the limit of a sequence is similar to what we are doing now. i.e., The limit of $\{f(x)\}$ is equal to $\lim\limits_{x \to \infty} f(x)$.

Example 7.5.7 *Find the limit of the sequence* $\left\{\frac{1}{x}\right\}$ *and* $\{x\}$.

The first one will be zero while the second one will be infinity.

When the limit is infinity, then we say it diverges. Oppositely we call it converges.

Comparing to the limit of a sequence, the sum of terms is concerned more frequently.

Definition 7.5.8 *A infinite sequence means the sequence has infinitely many terms.*

Definition 7.5.9 *A partial sum* S_n *is defined as* $\sum\limits_{n=0}^{\infty} a_n$.

We can obtain another series $\{S_n\}$ by the original sequences. Then the limit of $\{S_n\}$ will be equal to the limit of partial sum of a_n.

Definition 7.5.10 *We say the partial sum of* a_n *converges if* $\{S_n\}$ *has an finite limit. i.e.,* $\lim\limits_{n \to \infty} S_n$ *converges. Oppositely if the limit of* $\{S_n\}$ *diverges, then we say the sum of* a_n *diverges.*

Definition 7.5.11 *Infinite sum of* a_n *means* $\lim\limits_{n \to \infty} S_n$. *We denote it by S.*

Example 7.5.12 *Determine whether* $\left\{\frac{1}{2^n}\right\}$ *converges.*

We have $S = \lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \frac{1-0.5^n}{0.5} - 1 = 1$.

Therefore it converges to 1.

A very important note is that, the convergence of a sequence is not decided by the first finitely many terms (unless one of them is infinity), but it is decided by the later terms. So even $1+1+1+1+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$ converges, but $\frac{1}{100}+\frac{1}{101}+\frac{1}{102}...$ don't.

Now we will introduce a few method to determine the behavior of a sequence's infinite sum.

Theorem 7.5.13 *(n^{th} term test)If the limit of the sequence is not zero, then it diverges. However oppositely it may not valid. i.e., if the limit of the sequence is zero, it does not imply that the infinte sum of sequence converges.*

Proof: Let the limit of the given sequence be c. Then $S = \lim\limits_{n \to \infty} c + r = \infty$ where r is a constant. Thereofore it must diverge if the limit is not zero.

Example 7.5.14 $\{n\}$ *diverges since its limit is infinity.*

Example 7.5.15 $\left\{\frac{1}{n}\right\}$ *diverges, but its limit is zero.*

Example 7.5.16 $\left\{\frac{n-2}{n}\right\}$ *diverges since* $\lim_{n\to\infty} \frac{n-2}{n} = 1 \neq 0$.

Theorem 7.5.17 *(Geometric progression test)For sequences* $\{ar^n\}$*, it diverges if* $|r| \geq 1$ *and it converges if* $|r| < 1$.

Proof: It is quit obvious by considering the formula about sum of geometric sequence.

Example 7.5.18 $\left\{\frac{1}{2^n}\right\}$ *converges since the common ratio is* $0.5 < 1$.

Example 7.5.19 $\{1\}$ *diverges since the common ratio is* $1 \geq 1$.

Definition 7.5.20 *Zeta function is defined as* $\zeta(s) = \sum\limits_{n=1}^{\infty} n^{-s}$.

When the functino is used in convergence test. we only consider $s > 0$. However in analysis field it can be negative real, or even complex number. In fact, zeta function is the core idea od the **Riemann's Hypothesis** and we will talk about this later.

Theorem 7.5.21 *(Zeta functiion test) For* $\{\zeta(s)\}$*, it diverges if* $0 \leq s \leq 1$ *and converges if* $s > 1$.

The proof involves integration and it is listed below. However a basic idea can be deduced: The function is monotonically decreasing when $s > 1$. If $\zeta(1)$ is just enough to diverge, then the following converges.

The proof on zeta function test by integration

Lemma 7.5.22 *(Integration test) For* $\{f(x)\}$ *is ocntinous, has positive value for all* x *and monotonically deceasing for* $x \geq 1$*, then* $\int_1^{\infty} f(x)dx$ *and* $\sum\limits_{n=1}^{\infty} a_n$ *behaves the same. i.e., diverge together, or converge together.*

Proof is omitted. We can claim the zeta function test by this lemma. The proof is left for readers.

Example 7.5.23 $\left\{\frac{1}{n^3}\right\}$ *converges since* $3 > 1$.

Example 7.5.24 $\left\{\frac{1}{\sqrt{n}}\right\}$ *diverges since* $0.5 \leq 1$.

Example 7.5.25 *The harmonic series* $\left\{\frac{1}{n}\right\}$ *since* $1 \leq 1$.

Theorem 7.5.26 *(Comparason test) If all terms in* $\{a_n\}$ *and* $\{b_n\}$ *are positive and for all* n*,* $b_n \geq a_n$*, then if* $\{b_n\}$ *converges, then* $\{a_n\}$ *converges. If* $\{a_n\}$ *diverges, then* $\{b_n\}$ *diverges.*

Warning: If $\{b_n\}$ diverges, it does not imply that $\{a_n\}$ diverges. Similarly if $\{a_n\}$ converges, it does not imply that $\{a_n\}$ converges.

Proof: Define $c_n = b_n - a_n \geq 0$, then if $\{a_n\}$ diverges, then no matter what is $S_c > 0$, $S_a + S_c = \infty$. Similarly if b_n diverges, then $S_a = S_b - S_c$ which must give a constant, which implies that it converges.

Example 7.5.27 *Determine whether* $\left\{\frac{1}{7+2^n}\right\}$ *converges.*

Since all terms are positive, $\frac{1}{7+2^n} \leq \frac{1}{2^n}$ and $\left\{\frac{1}{2^n}\right\}$ diverges, therefore $\left\{\frac{1}{7+2^n}\right\}$ converges.

Example 7.5.28 *Determine whether* $\left\{\frac{1}{n+1}\right\}$ *converges.*

Method 1: $\frac{1}{n+1} \geq \frac{1}{2n}$ and $\left\{\frac{1}{2n}\right\}$ diverges, so $\left\{\frac{1}{n+1}\right\}$ diverges. Note that $\frac{1}{n}$ should not be used since $\frac{1}{n} \geq \frac{1}{n+1}$.

Method 2: $\sum\limits_{n=1}^{\infty} \frac{1}{n+1} = \sum\limits_{n=1}^{\infty} \frac{1}{n} - 1 = \infty$. So it diverges.

Theorem 7.5.29 *(Limit test) If all terms in* $\{a_n\}$ *and* $\{b_n\}$ *are positive and exist* k *such that* $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k > 0$*, then either they diverge together, or converge together.*

Basic idea on the proof: if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = k$, then $S_a \approx kS_b$. Mutiplying a constant will not affect the convergence, therefore the theorem is claimed.

Example 7.5.30 *Determine whether* $\left\{\frac{1}{\sqrt{n}+3}\right\}$ *converges.*

Note that comparason test is not valid since $\frac{1}{\sqrt{n}+3} \leq \frac{1}{\sqrt{n}}$.

By limit test, $\lim\limits_{n\to\infty} \frac{(\sqrt{n}+3)^{-1}}{(\sqrt{n})^{-1}} = 1$. Since $\left\{\frac{1}{\sqrt{n}}\right\}$ diverges, therefore $\left\{\frac{1}{\sqrt{n}+3}\right\}$ diverges.

Theorem 7.5.31 *If all terms of* $\{a_n\}$ *are positive,* $a_{n+1} \leq a_n$ *for all* n *and its limit is zero, then* $\{(-1)^n a_n\}$ *or* $\left\{(-1)^{n+1} a_n\right\}$ *converges.*

Example 7.5.32 *Determine whether* $\left\{(-1)^{n+1} n^{-1}\right\}$ *converges.*

It converges since $\frac{1}{n+1} \leq \frac{1}{n}$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and all terms are positive.

Theorem 7.5.33 *(Absolute convergence theorem) If* $\{|a_n\}$ *converges, then* $\{a_n\}$ *is absolutely converged.*

Note that if a sequence is absolutely converged, then it converges.

Theorem 7.5.34 *(Ratio test) For sequence $\{a_n\}$, if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the sequence is absolutely converging. If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then it diverges. If it equals to 1, then it tell us nothing.*

Example 7.5.35 *Determine whether $\left\{\frac{5^n}{n1}\right\}$ converges.*

By ratio test, $\lim\limits_{n\to\infty}\left|\frac{5^{n+1}n!}{5^n(n+1)!}\right| = \lim\limits_{n\to\infty}\frac{5}{n+1} = 0$. Therefore it converges.

Theorem 7.5.36 *For sequence $\{a_n\}$, if $\lim\limits_{n\to\infty}(a_n)^{\frac{1}{n}} < 1$, then it is absolutely converging. If it is larger than 1, then it diverges. If it equals to one, then it tell us nothing.*

Example 7.5.37 *Determine when does $\left\{\frac{x^n}{n!}\right\}$ converges.*

Method 1: By ratio test, $\lim\limits_{n\to\infty}\frac{x^{n+1}n!}{x^n(n+1)!} = 0 < 1$, therefore for any x it diverges.

Method 2: $S = e^x$, so for any given x is diverges.

Example 7.5.38 *Determine when does $\{n^n x^n\}$ converges.*

$\lim\limits_{n\to\infty}|n^n x^n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} n|x|$. It diverges as long as $x \neq 0$. So it converges only when $x = 0$. (Of course it converges at $x = 0$ since when $x = 0$ the whole series becomes zero.)

Exercise:

1. Finish the proof of theorem 7.5.20.
2. Evaluate $\lim\limits_{n\to 2}\frac{3x^2-12}{x^3-8}$.
3. Evaluate $\lim\limits_{n\to\infty}\frac{\tan x}{\sin x}$.
4. Evaluate $\lim\limits_{n\to 0}\frac{\sin x^2}{x}$.
5. Evaluate $\lim\limits_{x\to 0}\frac{e^x-x-1}{x^2}$.
6. Evaluate $\lim\limits_{x\to\infty} e^{-x}\ln x$.
7. Determine the limit of the following sequence: (i): $\{3^n - 2^n\}$, (ii): $\{7^{-n}\}$, (iii): $\{3n - 105\}$, $\{n^2 - 7\}$.
8. Show that when $P(x)$ is a polynomial where $\deg P \geq 1$, then $\{P(n)\}$ diverges.
9. Show that if a sequence diverges, then deleting finitely many terms will not affect the convergence.

10. Determine convergence of $\left\{\frac{n+2}{n+3}\right\}$, $\left\{\frac{2^n}{n^2}\right\}$, $\left\{\frac{n^2}{3n^2+1}\right\}$ and $\left\{\frac{\ln x^2}{x\ln x}\right\}$.

11. Determine the convergence of $\left\{\frac{1}{p_n}\right\}$, $\left\{\frac{1}{n^4}\right\}$ and $\left\{\frac{1}{\sqrt{n}}\right\}$.

12. Determine the convergence of $\left\{\frac{1}{2^n+3^n}\right\}$, $\left\{\frac{1}{n-\ln n}\right\}$, $\left\{\frac{1}{\sqrt{n}-1}\right\}$ and $\left\{\frac{1}{\sqrt{n}+1}\right\}$.

13. Determine the convergence of $\{(-3)^{-n}\}$, $\{(-n)^{-1}\}$, $\left\{\frac{(-1)^{n+1}n}{n+1}\right\}$ and $\left\{\frac{(-1)^n}{2^n}\right\}$.

14. Determine when does $\left\{\frac{nx^n}{3^n}\right\}$ and $\left\{\frac{\ln n^x}{n}\right\}$ converges.

15. Determine the convergence of $\left\{\frac{n!}{2^{n^2}}\right\}$, $\left\{\frac{\sqrt{n}}{n+1}\right\}$, $\left\{\frac{n!}{(2n)!}\right\}$ and $\left\{\frac{(n!)^2}{(2n)!}\right\}$.

16. Determine the convergence of $\left\{\frac{\prod(c_i n)!}{(kn)!}\right\}$ where $\sum c_i = k$.

Chapter 8

Equation(II)

8.1 System of equation

Example 8.1.1 *Given* $\sin x + \cos x = \frac{7}{5}$, *find* $\tan x + \cot x$.

Rather solving x, we better put it into the system of equation $\begin{cases} x^2 + y^2 = 1 \\ x + y = \frac{7}{5} \end{cases}$ and to find $\frac{x}{y} + \frac{y}{x} = \frac{x^2+y^2}{xy} = \frac{1}{xy}$.

Recall the elementary function, $xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2] = \frac{12}{25}$.

In this section we are going to study the skill of analyzing infomration and to make up conclusion without doing something too complicated.

The following is rather a complicated example.

Example 8.1.2 *Study the follow case.*

There are two scoring system in a game. Denote $C_1, C_2, ...$ be the combo number before the miss.

System A: Beside miss, the three scoring rank in each combo is divided into 300,100 and 50. Score in each combo is equal to $\sum_{i=1}^{C} Ria$, where R is the difficulty coefficient which is constant through the song and a is the scoring rank (300/100/50). Total score is equal to sum of each partial score.

System B: There are two scoring rank 300 and 100. Score in each combo is equal to $\sum_{i=1}^{n} f(i)a$. $f(x)$ is defined as $300 + 8i$ when $i \leq 100$ and it is equal to 1100 when i is larger than 100.

Now determine maximum score of combo C, and analysis the effect of missing once in some point, breaking C intwo two parts. Moreover, construct a bonding to describe the effect of conducting n misses, it can be in any distribution.

Highest scoring rank must be achieved if it is the maximum score. Then it equals to $150RC(C+1)$. The answer is in quadratic, so if you conducted one miss, then we will have $150R(C_1^2 + C_2^2 + C_1 + C_2)$. By considering $C_1 + C_2 = C$,

we will have $C_1^2 + C_2^2 + C_1 + C_2 < C^2 + C$. Let $C_1 \le C_2$, then $C_1(C_1+1) \le \frac{C}{4}(C+1) \le C_2(C_2+1) \le \frac{C}{2}(C+1)$. Summing up gives $150R(C_1^2+C_2^2+C_1+C_2) \le \frac{150RC(C+1)}{2}$. We can get $150R\sum_{i=1}^{n} C_i(C_i+1) \le \frac{150RC(C+1)}{n+1}$ by induction also. Equality holds when $C_1 = C_2 = ...$ Also, when there're one miss, the score is divided by a number between 1 and 2.

In system B, the maximum score is equal to $1100(C-100) + \sum_{i=1}^{100} f(i) = 1100C - 39600$ when $C \ge 100$. When the partial combo is larger than 100 each for each combo broken by the miss, the total score can be obtained by using this formula twice again. However, $1100 + 40400 = 41500$ points will be missed since $1100(100) > (110000 - 39600)$. Therefore when combo breaks, a constant of score will be reduced.

8.2 Roots of unity

Definition 8.2.1 *The n^{th} root of unity is the root of $z^n = 1$, where z is a complex number.*

Example 8.2.2 *The fourth roots of unity are 1, -1, i and $-i$.*

Definition 8.2.3 *A root of unity is called primitive if $z^k \neq 1$ where k is an integer that $0 < k < n$.*

Theorem 8.2.4 *A n^{th} roots of unity can be expressed in the forms of $e^{\frac{2k\pi i}{n}}$.*

Proof: $(e^{\frac{2\pi i}{n}})^k = e^{\frac{2k\pi i}{n}} \neq 1$ for $0 < k < n$ and $(e^{\frac{2\pi i}{n}})^n = (-1)^2 = 1$.

Lemma 8.2.5 *1 if any n^{th} roots of unity.*

Corollary 8.2.6 *The conjugate of a complex roots of unity is also a roots of unity.*

Definition 8.2.7 *We usually write one of the third roots of unity, $\frac{-1+\sqrt{3}i}{2}$ as ω.*

A certain properties can be shown form this definition, and they're listed as exercises.

Exercise:

1. Find all sixth roots of unity.
2. Think about the definition again. Express the roots of unity in tersms of trigonometric functions.
3. Show that the sum of roots of unity is zero.
4. Show that the product of roots of unity is 1.

8.3 Basic cubic equation

Proposition 8.3.1 *Substitute $x = y - \frac{a_2}{3}$ eliminates the x^2 term in $x^3 + a_2x^2 + a_1x + a_0 = 0$.*

It refers to proposition 3.1.4 again. It will be the ocre idea in solving basic forms of cubic equation. The original types refers to finding zeros of a polynomial of degree 3, without given any other information.

We will exmaine a few cases in this chapter about the original forms of cubic equation here, but before the discussion, we know know a fact clearly.

Proposition 8.3.2 *A real root MUST exist in cubic equation with real coefficients.*

A simple idea: assume the leasding coefficient is positive. When x is large enough, the polynomial is positive; when it is small enough (negative and large in magnitude), the polynomial is negative. Since polynomial is continous at every point, it must exist a zero.

When mean-value theorem and mid-point theorem is introduced later, a more deatailed proof can be given.

Example 8.3.3 *Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$.*

First we substitute $x = y + 2$, then it becomes $y^3 - y = y(y-1)(y+1) = 0$. $y = 0$, 1 or -1, therefore $x = 1$, 2 or 3.

Unluckily, most of the case iwll be much more complicated that we can't factorize them after the substitution.

Example 8.3.4 *Solve the equation $x^3 - 6x^2 + 9x - 4 = 0$.*

Substitute $x = y+2$ gives $y^3 - 3y - 2 = 0$. Now we let the solution be $u+v = y$. The equation becomes $(u+v)^3 - 3(u+v) - 2 = u^3 + v^3 + (u+v)(3uv-3) - 2 = 0$.

Since we can adjust u and v such that $3uv - 3 = 0$, i.e., $u^3v^3 = 1$, then $u^3 + v^3 - 2 = 0$. Note that in the two equations u and v is interchangable, therefore the solution is unique. (If the first solution is $u = a$ and $v = b$, then the second solution must be $u = b$ and $v = a$.) Then obviously we get the solution $u = v = 1$. Therefore one of the solution of y is 2.

Now divide $y^3 - 3y - 2 = 0$ be $(y-2)$ and get $y^2 + 2y + 1 = 0$ which gives a double root of $y = -1$. Add each roots by 2 and gives the root of x: 4 and double 1.

Example 8.3.5 *Solve the equation $x^3 - 6x^2 + 7x - 4 = 0$.*

Substitute $x = y + 2$ gives $y^3 - 5y + 2 = 0$.Substitude $y = u + v$ and rearranging gives $u^3 + v^3 + (u+v)(3uv - 5) + 2 = 0$. Then we will have two equation: $u^3v^3 = \frac{125}{27}$ and $u^3 + v^3 = -2$. Now we sovle the equation and get $u = 1 \pm \frac{\sqrt{6}i}{3}$. What does that mean? It means that u and v takes up the two values. While considering $u+v$, the imaginary part is eliminated and the answer

is 2. Divide $y^3 - 5y + 2 = 0$ by $(y-2)$ gives the remaining root $(\sqrt{2}-1)$ and $(\sqrt{2}-1)$. Add 2 back gives the root of x: 4, $(1+\sqrt{2})$ and $(1-\sqrt{2})$.

However it is extremely hard so that we will discuss the previous example now.

Example 8.3.6 *Solve the equation* $\left\{\begin{array}{l} x^3 + y^3 = -2 \\ x^3y^3 = \frac{125}{27} \end{array}\right.$

Note that u and v is interchangable in the equation so that the two sets of solution is in fact the same. Some simple tricks can help to get $u^3 = -1 \pm \sqrt{\frac{98}{27}}i = -1 \pm \frac{7}{3}\sqrt{\frac{2}{3}}$.

Lemma 8.3.7 *Result of addition or multiplication among algebraic numbers is still algebraic numbers.*

This is the **closure** of algebraic numbers. Therefore we can express $\sqrt[3]{-1+\frac{7}{3}\sqrt{\frac{2}{3}}}$ in terms of $a + b\sqrt{\frac{3}{2}}$.

$$(a+b\sqrt{\frac{3}{2}})^3 = a(a^2-2b^2) + \sqrt{\frac{3}{2}}bi(3a^2 - \frac{2b^2}{3}) = -1 + \frac{7}{3}\sqrt{\frac{2}{3}}i$$

By comparing the real and imaginary part and after some elimination, we will have the equation $\left\{\begin{array}{l} a(a^2-2b^2) = -1 \\ b(3a^2 - \frac{2b^2}{3}) = \frac{7}{3} \end{array}\right.$. It would convinence us if both variables are 1. Therefore $\sqrt[3]{-1+\frac{7}{3}\sqrt{\frac{2}{3}}} = 1 + \sqrt{\frac{3}{2}}$.

It will be more complicated if the remaining two roots is needed. However once we find one root, we can already find all three roots of the original equation.

Example 8.3.8 *Solve the equation* $x^3 + 6x^2 + 10x + 8 = 0$.

Substitute $x = y - 2$ gives $y^3 - 2y + 4 = 0$, substitude $u + v = y$ gives $u^3 + v^3 + (u+v)(3uv-2) + 4 = 0$. By solving the equation $\left\{\begin{array}{l} u^3 + v^3 = -4 \\ u^3v^3 = \frac{8}{27} \end{array}\right.$ and get $u = -2 \pm \frac{10}{3}\sqrt{\frac{1}{3}}$, we get another equation $\left\{\begin{array}{l} a^3 + ab^2 = -2 \\ 3a^2b + \frac{b^3}{3} = \frac{10}{3} \end{array}\right.$. We can get an obvious solution $u = -1 \pm \sqrt{\frac{1}{3}}$, therefore $y = -2$ is one of the solution. Therefore $x = -4$ is a solution. Divide $x^3 + 6x^2 + 10x + 8$ by $(x+4)$ and get $x^2 + 2x + 2$ which gives roots $-1 \pm i$.

A quick note on the method of solution:

You may ask that why u and v is interchangable implies that the solution in unique. We may think in this way: If $f(u,v)$ is symmetical, then the graph of $f(u,v)$ is also symmetrical along the line $u = v$. Therefore if (u,v) is a solution, then (v,u) is a solution too. We say unique because we find u^3 and v^3, then

directly take their cubic root. We don't need to find the remaining four root and these two is identical if we add them together.

Also, when we obtain the equation about u^3 and v^3, it can be very hard when we try to find even one solution only. But if we can find u^3 in surd form, we can express u in similar form also.

Exercise:

1. Show that $\sqrt[3]{a+bs} = x + yz$, then $\sqrt[3]{a-bs} = x - ys$. Hence show that the equation about u and v has unique pair of solution. i.e., if $u = a \pm b$, then $v = a \mp b$.

2. Solve the equation $x^3 + 3x^2 - 6y - 36 = 0$.

3. Solve the equation $x^3 - 3x^2 - 38x + 60 = 0$.

4. Solve the equation $x^3 + 3x^2 - 3x - 1 = 0$.

When answers are not required to express in surd form, calculations can be easier with the help of trrigonometry and calculators.

First of all, when we acquire the equation of u and v, we should aware that all roots can be the solution of y. Moreover when we acquire the solutions of u and v, they must form exactly three pairs of roots. It is because when a is a solution of $u^3 = c$, then ωa and $\omega^2 a$ is also. When there're three real roots, we can use the following method:

After solving the equation about u and v, we let $u^3 = (x)cisr$. Obviously $u = \sqrt[3]{x}cis\frac{r}{3}$. However ωu and $\omega^2 u$ is the root also. Similarly v will be the conjugate of the three roots of u. Then when we considering $u + v$, it equals to $2\sqrt[3]{x}Re(u)$, $2\sqrt[3]{x}Re(\omega u)$ and $2\sqrt[3]{x}Re(\omega^2 u)$.,

Example 8.3.9 *Solve the equation $x^3 - 3x - 1 = 0$.*

By putting the same substitution, we will have $u^3 = \frac{1+\sqrt{3}}{2} = cis60°$ as one of the solution. Then by De Moivre's formula, $u = cis20°$. Since $\omega = cis120°$, therefore three solutions are $2\cos 20°$, $2\cos 140°$ and $2\cos 260°$.

8.4 Dsicriminant and natrue of roots

Theorem 8.4.1 *If a_1, a_2,...a_n are roots of a polynomial of degree n, then $\prod\limits_{i>j}(x_i - x_j)^2$ is the discriminant of the polynomial.*

Example 8.4.2 *Let x_1 and x_2 be roots of a quadratic polynomial $^2 + bx + c = 0$. Then $(x_1 - x_2)2 = (x_1 + x_2)^2 - 4x_1x_2 = b^2 - 4c$. Which is the discriminant of quadratic equation when $a = 0$.*

Similarly, the discriminant of a cubic polynomial will be $[(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)]^2$ where x_i are roots of the polynomial. However it will be too complicated when we expand it in terms of coefficient of polynomial. But if it is

in forms of x^3+ax+b, then it will be much easier. Obviously we can transform it into that form. The discriminant $\Delta = -4a^3 - 27b^2$. The proof is left for readers.

Theorem 8.4.3 *If $\Delta > 0$, then it will have three distinct real roots. If $\Delta < 0$, then it has a pair of complex roots (one is conjugate of another) and a real root. If $\Delta = 0$, then it contains multiple roots while all roots are real.*

Proof: When all roots are real, obviously $\Delta > 0$. If it has multiple root, then one of the term becomes zero and then $\Delta = 0$. If there're complex roots, let $x_1, x_2 = a \pm bi$ be the complex roots while $x_3 = c$ is the real root.

$\Delta = [(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)]^2 = [(2bi)(a - c + bi)(a + c + bi)]^2$, since $[(a - c + bi)(a + c + bi)]^2]$ is a real number, therefore Δ is negative due to the existance of i.

Example 8.4.4 *Find all x such that $x^3 - 3x^2 + k = 0$ has three distinct real roots.*

Substitute $x = y+1$ gives $y^3 - 3y + (k-2) = 0$. $\Delta = -4(-3)^3 - 27(k-2)^2 = 27(4 - (k-2)^2) > 0$, $|k - 2| < 2$ gives $4 > k > 0$.

Example 8.4.5 *Given $x^3 - 27x + q = 0$ has a pair of repeated root, find q*

Method I: Differentiation: $(x^3 - 27x + q)' = 3x^2 - 27$, solve $3x^2 - 27 = 0$ and get the multiple root ± 3. By substitution we get $q = \pm 54$.

Method II: By discriminant: $\Delta = -4(-27)^3 - 27q^2 = 0$, $4(27)^2 = q^2$, $q = \pm 54$.

Note that both $x^3 - 27x - 54 = 0$ and $x^3 - 27x + 54 = 0$ has a multiple root.

8.5 Quartic equation

Lemma 8.5.1 *A pair of complex solution $a + bi$ and $a - bi$ comes from the equation (factor) $x^2 - 2ax + a^2 + b^2 = 0$ and hence, every polynomial with real coefficients can factorized into quadratic factor with real coefficients.*

The proof is left for readers.

Example 8.5.2 *Solve the equation $x^4 - 8x^3 + 17x^2 + 2x - 24 = 0$.*

Apply the same elimination we get $y^4 - 7y^2 + 6y = 0$. Now don't be hurry to take away the common factor y, notice that when it is factorized into two quadratic real factor, the coefficient of x term has the same magnitude (since the x^3 term is eliminated). Therefore we written the equation as $(x^2 + px + s)(x^2 - px + r) = 0$. Then we will get three equations: (1)(x^2) $r + s - p^2 = -7$ (2)(x) $p(r - s) = 6$ (3) $rs = 0$. Since r and s is interchangable while changing the sign of p: $(-p, r, s)$ or (p, s, r), we WLOG (without loss of generality) let $r = 0$. The equation becomes (1) $s = p^2 - 7$ and (2) $s = \frac{-6}{p}$. Then we will have

$p^3 - 7p + 6 = (p-2)(p-1)(p+3) = 0$, Why do we get three solutions? Let's try to substitude them:

Case 1: p=2, then $s = -3$ and the equation becomes $(y^2+2y-3)(y^2-2y) = y(y-1)(y+3)(y-2) = 0$.

Case 2: p=1, then $s = -6$ and the equation becomes $(y^2+y-6)(y^2-y) = y(y-1)(y-2)(y+3) = 0$.

Case 3: p=-3, then $s = 2$ and the equation becomes $(y^2-3y+2)(y^2+3y) = y(y-1)(y-2)(y+3) = 0$.

Therefore we see that the solutions of p is unique to the roots of y. Therefore we get the root of x: -1,2,3, and 4.

As a summary we know that solving a n^{th} degree equations is relying on the method to solve the previous degree's equation. For example, we need the factor theorem (linear) to support our general solution to quadratic equation, and we use the ides of sum and product of roots to resolve $(u+v) = y$ for cubic equation. Lastly, we have to solve a cubic equation during quartic equation.

So, can be solve the quintic equation? Any general solution for quintic equations?

The answer is yes for the first question while the second one is NO. It refers to an extremely difficult proof so it is omitted. Each eqution can be solve distinctly, bur no general solution exist. Moreover, some equations such as $x^5 - 58x + 17 = 0$ can't be represented by surd forms.

Exercise:

1. Based on the last paragraph, some roots can't be expressed in surd forms. Are they algebraic?

2. Show that real or complex roots exist in pairs for quartic equation, both graphically and albegrically.

3. Solve the equation $x^4 + 6x^3 + 18x^2 + 30x + 25 = 0$.

4. Solve the equation $x^4 + 5x^3 - 30x^2 - 40x + 64 = 0$.

5. Show that equation in forms of $(x-a-b)(x-a+b)(x+a-b)(x+a+b) = 0$ eliminates the x and x^3 term after elimination. Show that it is valid even b is a complex number. Hence solve the equation $x^4 + 6x^2 + 25 = 0$ and $x^4 - 6x^2 + 1 = 0$.

6. Solve the equation $4x^4 - 16x^3 + 20x - 9 = 0$.

8.6 Special equation about polynomial

Example 8.6.1 *Given a equation $x^3 + 15x^2 + 49x + 45 = 0$ gives roots which can be arranged into arithmetic sequence with common difference 4. Without solving the equation, find the smallest root. (smallest in magnitude)*

Let the three root be $k-4$, k and $k+4$ respectively. By Viete's theorem, we have $\sigma_1 = 3k = -15$, $k = -5$. Then the smallest root will be -5+4=1.

Example 8.6.2 *(8.5 exercise revisited) Given equation* $x^4+5x^3-30x^2-40x+64=0$ *gives roots which form a geometric sequence with common ratio -2. Without solving the equation find the largest root. (Largest in magnitude)*

Let the three root be x, $-2x$, $4x$ and $-8x$ respectively. Then by Viete's theorem, $\sigma_1=-5x=-5$, $x=1$, then the largest root will be -8.

Example 8.6.3 *Given equation* $35x^3-33x^2-3x+1=0$ *gives roots which form a harmonic sequence (i.e.,* $\frac{1}{x-r}$, $\frac{1}{x}$ *and* $\frac{1}{x+r}$*). Without solving the equation, find all the roots.*

Firstly we change the equation into $x^3-\frac{33}{35}x^2-\frac{3}{35}x+\frac{1}{35}=0$, Let the roots be $\frac{1}{x-r}$, $\frac{1}{x}$ and $\frac{1}{x+r}$ respectively. By Viete's theorem, we have $\sigma_1=\frac{3x^2-r^2}{x(x^2-r^2)}=\frac{33}{35}$ and $\sigma_3=\frac{1}{x(x^2-r^2)}=\frac{-1}{35}$. Rewriting the equation gives $3x^2-r^2=-33$ and $x^2-r^2=\frac{-35}{x}$. Substitute the second one into the first one gives $2x^2-\frac{35}{x}=-33$ which gives an obvious solution 1. Then $r=6$ which gives the root: $\frac{-1}{5}$, 1 and $\frac{1}{7}$.

Note that a cubic equation is involved, while the remaining two roots are complex roots. Readers may try that whether the complex answer valid for the equations.

Just like those palindromic numbers, polynomials can be palindromic as well.

Theorem 8.6.4 *If a polynomial can be factorized in forms of* $\prod(x-r_i)(x-\frac{1}{r_i})$*, then the expanded result will be palindromic for all polynomial with even power.*

Proof: Firstly we show that it is true for quadratic polynomial: $(x-a)(x-\frac{1}{a})=x^2-(a+\frac{1}{a})x+1=0$ and it is true.

Now we show that palindromic polynomial of even power **obeys closure with multiplication**, i.e., the product between two parlindromic polynomial is still parlindromic.

Consider two paradromic polynomial, one with degree $2n$ and another is quadratic. Denote the coefficient of x^n term as c_n while the qradratic one is (x^2+ax+1). Denote the coefficient of x^n of the product as d_n, then we will have $c_i=c_{2n-i}$. For any $0<i<2n+2$, we have $d_i=c_i+c_{i-2}+ac_{i-1}=c_{2n-1}+c_{2n+2}+ac_{2n+1}=d_{2n+2-i}$. The closure is confirmed as the claim.

However from the proof, we know that complex root coming from parlindromic qeuadratic polynomial doesn't obey the rule.

Example 8.6.5 *Solve the equation* $4x^4+9x^3+26x^2+9x+4=0$.

Let $x+x^{-1}=y$, then $y^2-2=x^2+x^{-2}$ and $y^3-3y=x^3+x^{-3}$.

The equation is equivalent to $x^2(4x^2+9x-26+9x^{-1}+4x^{-2})=x^2[4(y^2-2)+9y-26]=0$, $4y^2+9y-34=0$. By solving $x+x^{-1}=2$ or $\frac{-17}{4}$, we get the root 1 (repeated), 4 and $\frac{-17}{4}$.

Example 8.6.6 *Let* $f(x)=x^6+ax^5+bx^4+cx^3+bx^2+ax+1$*, where* a*,* b *and* c *are real numbers. Suppose* μ *is a repeated root of* f*, show that* μ^{-1} *is also a root*

of f. Hence find the repeated root for $4x^6-16x^5+17x^4-7x^3+17x^2-16x+4=0$, and determine whether it can be factorized into linear polynomials with real coefficients. (modified AL 2010)

Suppose $f(x)=(x-p)^2P(x)$. By theorem 8.6.4, $(x-\frac{1}{p})^2$ must be a factor of $f(x)$. For the equation, we can find the repeated roots are 2 and $\frac{1}{2}$, so that by dividing the polynomial by $(x-2)^2(x-\frac{1}{2})^2$ and get $4(x^2+x+1)$, we can obtain the answer (no) by checking the discriminant.

Example 8.6.7 *Given the equation $x^4+10x^3+15x^2-50x-56=0$ has four real roots, forming an arithmetic sequence. Find the roots.*

Let the four roots be $(x-r)$, x, $(x+r)$ and $(x+2r)$ respectively. Check: $\sigma_1=4x+2r=-10$ and $\sigma_2=6x^2+6xr-r^2=15$. By solving the equation We get the solution $(x,r)=(-4,3)$ or $(1,-3)$. (Readers can check that they are actually the same. Also the magnitude of difference must be the same.) The four roots are -7,-4,-1 and 2.

Exercise:

1. Given the equation $x^4-4x^3+46x^2-84x+185=0$ gives roots which forms arithmetic sequence. Without solving the equation find the roots.
2. Given the equation $3x^4-40x^3+130x^2-120x+27=0$ gives roots which forms geometric sequence. Without solving the equation find the roots.
3. Given the equation $15x^4-8x^3-14x^2+8x-1=0$ gives roots which forms harmonic sequence. Without solving the equation find the roots.
4. Given the equation $x^4-17x^3+53x^2+185x-462=0$ gives roots which the magnitude of them are co-prime each other. Without solving the equation find the roots.
5. Find the unique characteristic of palindromic polynomial of odd degrees and prove it.
6. Is it all palindromic polynomial obeys closure?
7. Solve the equation $7x^4-36x^3-86x^2-36x+7=0$,
8. Solve the equation $14x^5-121x^4+143x^3+143x^2-121x+14=0$.
9. Solve the equation $50x^6-395x^5+202x^4+2590x^3+20x^2-395x+50=0$. (Difficult)

8.7 Application: integer function and Riemann's hypothesis

Example 8.7.1 *Find the range of solution of $[x]^2=[x^2]$ for x is a positive real number.*

From the equation, we know that both sides must be integer. When $x^2 \geq [x]^2+1$, equality will not hold. Therefore we have 0 to 1, 1 to $(\sqrt{2}-1)$, 2 to $(\sqrt{5}-2)$,.... The sum of range, that is $\sum\limits_{i=0}^{\infty}(\sqrt{x^2+1}-x)$.

However, we know that the sum is diverging. Why?

Let $\sqrt{x^2+1}-x=k$. Then $(x+k)^2=x^2+1$, and $k^2+2kx-1=0$. When x is large, k tends to zero, and we can now bond k:

Let a number $k_1 < k$ such that $3kx-1=0$. It is valid as long as $k<1$ as this is valid for all k. Now we have $k=\frac{1}{3x}$. Since $\sum\limits_{i=0}^{\infty}(\sqrt{x^2+1}-x) > \sum\limits_{i=0}^{\infty}\frac{1}{3x} = \frac{1}{3}\sum\limits_{i=0}^{\infty}\frac{1}{x}$ which diverges. Therefore the original summation diverges.

Now we extend the domain of zeta function to complex numbers. Some properties can be found in this function.

Firstly we can find that when the real part of $\zeta(c+it)$, where c is a fixed real constant, is going up and down while the mean in creasing like a power function. In order to describe the case, we have to use the big-O function.

Definition 8.7.2 *We say $f(x)=O(g(x))$ if and only if there exist a positive real number M and a real number x_0 such that $|f(x)| \leq M|g(x)|$ for all $x > x_0$.*

When two functions can be described as $f(x)=O(g(x))$, then rate of increasing will be moer or less the same, difference in a constant muliple only.

Example 8.7.3 $f(x)=6x^2+1=O(x^2)$ *since* $|f(x)| \leq 7|x^2|$ *for all* $x \geq 1$.

Now we are going to the core analysis of this issue: introducing the Riemann's hypothesis and show that $f(x+1)=O(\zeta(x))$ where $f(x)=\sum\limits_{i=0}^{\infty}(\sqrt[x]{i^x+1}-i)$.

Theorem 8.7.4 *The identity, proved by Euler, states that $\zeta(s)=\sum n^{-s}=\prod(1-p^{-s})^{-1}$.*

Euler's proof:

$$\zeta(s)=1+2^{-s}+3^{-s}+4^{-s}+... \Leftrightarrow 2^{-s}\zeta(s)=2^{-s}+4^{-s}+6^{-s}+8^{-s}+...$$

$$(1-2^{-s})\zeta(s)=1+3^{-s}+5^{-s}+7^{-s}+9^{-s}+...$$

$$(1-3^{-s})(1-2^{-s})\zeta(s)=1+5^{-s}+7^{-s}+11^{-s}+...$$

Repeating the same procedure we get $\prod(1-p^{-s})\zeta(s)=1$, giving the required identity.

Now we try to show that $f(x+1)=O(\zeta(x))$.

Apply the same approach of $f(2)$, we know that $\sqrt[x]{i^x+1}-i=k$ is equivalent to $\sum\limits_{t=0}^{x-1}C_t^x i^t k^{x-t}-1=0$. As long as $i>1$, we will have a smaller $k_1<k$ so

that all terms is turned into $i^{x-1}k_1$, and $2^{x-1}i^{x-1}k_1 = -1$ and we will have $k_1 = \frac{1}{2^{x-1}i^{x-1}}$.

Therefore $\sum\limits_{i=0}^{\infty} \sqrt[x]{i^x+1} - i < \sum k_1 = \sum\limits_{i=1}^{\infty} \frac{1}{2^{x-1}i^{x-1}} = \frac{1}{2^{x-1}} \sum\limits_{i=1}^{\infty} \frac{1}{i^{x-1}} = \frac{1}{2^{x-1}}\zeta(x-1)$

Since $f(x+1) < \zeta(x)$ for all non-negative x, therefore $f(x+1) = O(g(x))$ is valid.

The divergence of $f(x)$ is shown also since $\zeta(1) = \lim\limits_{x\to\infty} H_x$ which diverges. We also know that $f(x)$ converges for all $x > 2$ since $\zeta(s)$ converges for all $x > 1$ and f is bounded below the zeta function by big-O notation.

Now we will discuss more example on big-O notation, mainly related to the Riemann's hypothesis.

Theorem 8.7.5 *The Prime Number Theorem states that $\pi(x)\ \frac{x}{\ln x}$, where $\pi(x)$ denotes number of primes under x and means the proportion between them infinitely close to 1.*

The Prime Number Theorem is first proposed by Gauss and later proved by Euler.

Theorem 8.7.6 *A better version of the Prime Number Theorem is $\pi(x)\ Li(x)$, where $Li(x) = \int_0^x \frac{dr}{\ln t}$. Moreover, $\pi(x) = Li(x) + O(\sqrt{x}\ln x)$.*

This reslut is proposed by Helge von Koch in 1901.

Definition 8.7.7 *Riemann hypothesis is a open problem that the non-trivial zeros of the zeta function has real part $\frac{1}{2}$.*

Once it is proposed by Riemann, a great mathematician, it has been an important issue. A weaker form of the hypothesis is the Lindelof's hypothesis.

Definition 8.7.8 *We define Lindelof's function $\mu(\sigma) = x$ if $|\zeta(\sigma + it)| = O(\mu(\sigma))$.*

It is know that $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leq 0$ and it is identically zero for $\sigma \geq 1$. Also it is known that this function is convex. The hypthesis is that $\mu(\sigma) = |\frac{1}{4} - \frac{\sigma}{2}| + \frac{1}{4} - \frac{\sigma}{2}$. (Readers can try to draw this graph. This hypothesis is weaker than Riemann's hypothesis. If Riemann's hypothesis is true, then this is true. However if Lindolf's hypothesis is true, it doesn't implies that Riemann's hypothesis is true.

Note that $f(x) = O(1)$ doesn't imply that it has zeros. An obvious case will be $f(x) = 1$.

Up to now a basic frame of algebraic skills should be given, and we can solidify our skills by the last chapter, inequality.

Chapter 9

Inequality

9.1 Nature of inequality

Definition 9.1.1 *An inequaliy $a > b$ implies that $a = b+c$, where c is a positive number.*

We should have encountered different inequality signs before, but we will discuss the nature of inequality here, as well as proving and solving inequality.

Definition 9.1.2 *An inequality $a \geq b$ implies that $a = b + c$, where c is a non-negative number.*

Definition 9.1.3 *An inequality $a \neq b$ implies that $a = b + c$, where c is non-zero.*

Basically they obey most of the numerical rules except one: if $a \geq b$, then $-a \leq -b$.

Proof: Let $a = b + c$, where c is non-negative. Then $-a = -b - c$ and $-b = -a + c$.

Example 9.1.4 $2 > 1$ *since* $2 = 1 + 1$ *and* 1 *is positive.*

I managed to give a basic frame of inequality in this section, they will be useful in proving inequality.

Exercise:

1. Show that if $a > b$ and c is a constant, then $a + c > b + c$.
2. Show that if $a > b$ and $c > d$, then $a + c > b + d$.
3. Show that if $a > b$ and $c > 0$, then $ac > bc$.
4. Show that the inequalities hold even $>$ is replaced by $\geq$.
5. Determine when will the above inequalites hold when $>$ is replaced by $\neq$.

6. Determine when will the above inequalities hold when they are complex numbers.

7. Show that if $a \geq b$ and $b \geq a$ at the same time, then $a = b$.

9.2 Solving inequality

Lemma 9.2.1 *Rearrangements of terms in inequality just like rearrangements of terms in equation.*

Proof:

1. If $x+a > b$, then $x = a-b$ since there exist a positive c so that $x+a = b+c$ and $x = (b-a)+c$, therefore $x > b-a$.

2. If $ax > b$, then $x > \frac{b}{a}$ if $a \geq 0$. It is bevause there exist a positive c so that $ax = b+c$, rearrangement gives $x = \frac{b}{a} + \frac{c}{a}$. Therefore $x > \frac{b}{a}$ since $\frac{c}{a} > 0$. (Note: $x > \frac{c}{a}$?)

Example 9.2.2 *Solve the inequality* $x + 2 > 5$.

We have $x > -3$.

Example 9.2.3 *Solve the ineequality* $\tan x > \sqrt{3}$ *for* $0 \leq x \leq 90°$.

We have $60° < x \leq 90°$.
Now we will introduce some notation:

1. If x is a real number that $a < x < b$, then $x \in (a, b)$.

2. If x is a real number that $a < x \leq b$, then $x \in (a, b]$.

3. if x is a real number that $a \leq x < b$, then $x \in [a, b)$;

4. If x is a real number that $a \leq x \leq b$, then $x \in [a, b]$.

5. If $x > a$, then $x \in (+\infty, a)$.

6. If $x < b$, then $x \in (b, -\infty)$.

The sign $\in$ means that x is in the range (set) of that numbers. () implies close interval and [] implies open interval.

Example 9.2.4 *Solve the inequality* $x + 4 > 7$.

We can apply the above notation: $(x+4) \in (7, +\infty)$, then $x \in (3, +\infty)$ since $7 - 4 = 3$ and $+\infty - 4$ is still $+\infty$.

Theorem 9.2.5 *Given a inequality $f(x) > g(x)$, we solve $f(x) = g(x)$ and get the roots $a_1, a_2, ...$. If some of the roots satisfy $f'(a_i) = g'(a_i)$, take away that root. Rearrange it in order $b_1 < b_2 < ....$ The answer is either $(-\infty, b_1)$, (b_2, b_3),... (b_{2n}, b_{2n+1}), OR (b_1, b_2), (b_3, b_4),..., (b_{2n-1}, b_{2b}),.... Note that the last interval will be $(b_j, +\infty)$.*

Proof: Let $f(a_1) > g(a_1)$. Then after intersection, $g(x) > f(x)$ until the next intersection. However, the change of $sgn(g(x) - f(x))$ won't change if $f'(x) = g'(x)$, that means two lines only touches. They won't cross over each other if and only if their tangent are equal.

Example 9.2.6 *Solve the inequality $x^2 - 3 > 1$.*

Method I: We have $x^2 - 4 = (x-2)(x+2) > 0$. If $x > 2$, then it is positive. If $x < -2$, then it's positive also. If $x \in [-2, 2]$, then it will be negative. Therefore the answer is $(-\infty, -2)$ and $(2, +\infty)$

Method II: Solve $x^2 - 4 = 0$ and get $x = 3$ or -2. Since $x^2 - 4 > 0$ for $x > 2$, it is also valid for $x < -2$..

Example 9.2.7 *What can be deduced from the given inequality $xf(x) \geq f(x)$?*

Let the roots of $f(x)$ be a_i. Then $x \geq 1$. Therefore the dmoain of $f(x)$ must be a subset of $[-1, +\infty)$.

Example 9.2.8 $\begin{cases} xy > 20 \\ x > 5 \end{cases}$

Since $x > 5 > 0$, we have $y > 5$.

Example 9.2.9 $\begin{cases} xy > 20 \\ x > -2 \end{cases}$

For $x \in (0, +\infty)$ and $xy \in (20, +\infty)$, $y \in (0, +\infty)$. it approach to zero when x large, and approach to infinity when x approach to zero.

For $x \in (-2, 0)$, $y \in (-\infty, -10)$. These two ranges will be the solution of y.

Note that if $x > a$ and $y < b$, we don't know whether xy or ab is bigger.

Example 9.2.10 $\begin{cases} xy \geq 20 \\ x + y \leq 9 \end{cases}$

From (2) we have $x \leq 9 - y$ and substitute this into (1): $y(9-y) \geq xy \geq 20$, $(y-4)(y-5) \leq 0$ and $y \in [4, 5]$. Correspondingly $x \in [4, 5]$. However we have to consider the negative case also. $y \in (-\infty, 0)$ and therefore $x \in (-\infty, 0)$.

Example 9.2.11 *Solve the inequality $\frac{x+3}{x+7} > 1$.*

We don't know whether the sign change as we multiply $(x+7)$, so we multiply $(x+7)^2$ to both sides: $(x+3)(x+7) \geq (x+7)^2$ and get $x < -7$.

Exercise:

1. Solve the inequality $3x + 8 > 9$.
2. Solve the inequality $5x - 3 > 28$.
3. Solve the inequality $x^2 - 5x + 6 < 0$.
4. Solve the inequality $x^3 - 3x^2 - 13x + 15 = 0$.
5. Solve the inequality $\begin{cases} xy > 45 \\ x^2 + y > 13 \end{cases}$.
6. Solve the inequality $\begin{cases} x^2 + y > 20 \\ x + 2y > 12 \end{cases}$.
7. Solve the inequality $\begin{cases} xy < 16 \\ x + y > 7 \end{cases}$.
8. What if the above inequality's sign changed to $\geq$ or $\leq$?
9. What if the above inequality's sign changed to neq?
10. Solve the inequality $\frac{2x+3}{x+3} \leq 2$.
11. Solve the inequality $\frac{(x+1)(x+2)}{x+3} \geq x + 4$.
12. Solve the inequality $\frac{x+1}{x+2} \geq \frac{x+3}{x+2}$.

Example 9.2.12 *Solve the inequality $|x| < 1$.*

By spliting case we get $x \in (-1, 1)$.
In fact, absolute value function always appear in the inequalities. For example, $|x| \geq 0$. Now let's discuss the graph about absolute value function.

Lemma 9.2.13 *The graph $y = |x + c|$ is a translation of $y = |x|$, where the graph is moved left by c.*

Firstly we define how to move the graph. For every point $(x, f(x)$ in the original graph is transformed to $(g(x), h(f(x)))$. If g and h is linear, then we said that it is a traslation, which the shape is remained same. In this example we have every point $x, |x|$ translated to $(x - c, |x|)$ since the magnitute of slope of the graph is 1.

Example 9.2.14 *Determine number of solutions of $|x - 2| = k$, where k is a positive number.*

For $|x| = k$ we have 2 solution, and when the graph is translated horizontally, both roots move 2 units towards right, while number of roots remains the same.

Example 9.2.15 *Determine number of solutions of $|x - |x - 2|| = 2$.*

By spliting case we have $|x - |x - 2|| = 2$ when $x \in [2, +\infty)$ and $|x - |x - 2|| = 2x - 2$ when $x \in (-\infty, 2)$. That gives the range $[2, +\infty)$ as solution.

9.3 Proving an inequality

Example 9.3.1 *Show that* $f(x+1) < \zeta(x)$.

$f(x)$ is the function defined at the last of chapter 8. This result will be obvious when we compare each term.

$$\sqrt[i+1]{k^{i+1}+1} - k < k^{-i} \Leftrightarrow \sqrt[i+1]{k^{i+1}+1} - k < k^{-i} + k$$

$$k^{i+1} + 1 < (k^{-i} + k)^{i+1}$$

is enough to show the claim.

In this section, we start to prove that some inequality is valid for a wide range of number.

Lemma 9.3.2 $a^2 \geq 0$ *for all real number.*

This is obvious enough, but how can we apply this inequality?

Lemma 9.3.3 $a^2 + b^2 \geq 2ab$ *for all real number.*

If $sgn(ab) = -1$, then it is no need for us to prove since $a^2 + b^2 \geq 0 > ab$. If $ab \geq 0$, then the original inequality is equvalent to $a^2 - 2ab + b^2 = (a-b)^2 > 0$.

Another method of proving inequality is by definition, but it may be too complicated for some problems.

Example 9.3.4 *Show that* $a^2 + b^2 \geq 2ab$ *by definition.*

Since the expression is symmetric, we WLOG let $a \geq b$ while $a = b + c$.

The expression becomes $a^2 + (a+c)^2 = 2a^2 + 2ac + c^2 = 2a(a+c) + c^2 = 2ab + c^2 \geq 2ab$. Moreover we can claim that equality holds only when $c = 0$, i.e., $a = b$.

Example 9.3.5 *Find the maximum of* ab *for* a, b *are positive real with* $a^2+b^2 = 1$.

We have $b = (1-a)$, $1 = a^2 + b^2 \geq 2ab$, so the maximum of ab will be $\frac{1}{2}$.

From now on, we will use more notation on cyclic sum and elementary function.

Example 9.3.6 *Show that* $\sum\limits_{cyc} x^2 \geq \sum\limits_{cyc} xy$ *for* $x, y, z \geq 0$.

We have $\sum\limits_{cyc}(x^2 - xy) = \frac{1}{2}\sum\limits_{cyc}(2x^2 - 2xy) = \frac{1}{2}\sum\limits_{cyc}(x^2 - 2xy + y^2) = \frac{1}{2}\sum\limits_{cyc}(x-y)^2 \geq 0$. Note that $\sum\limits_{cyc}(x^2 - 2xy + y^2) = \sum\limits_{cyc}(2x^2 - 2xy)$ is clearly shown in the expansion result.

Example 9.3.7 *Let a, b, c, d be any positve real. Find the the range of* $S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$. *(IMO 1974)*

By changing the denominator we will have:

$$S < \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{c+d} + \frac{d}{c+d} = 2$$

$$S > \sum \frac{a}{a+b+c+d} = \frac{\sigma_1}{\sigma_1} = 1$$

Therefore $1 < S < 2$. (However the proof about $S \in (1, 2)$ is omitted here but it is nesscary to complete this problem.)

Sometimes we need an inductino to prove an inequality.

Example 9.3.8 *Show that $a^3 + b^3 + c^3 \geq 3\sigma_3$ for $a, b, c \geq 0$.*

We have $2(a^3 + b^3 + c^3 - 3abc) = 2(a+b+c)(a^2 + b^2 + c^2 - ab - ac - ca) = \sigma_1 \sum\limits_{cyc}(a-b)^2 \geq 0$.

As a generalization, we will have the AM-GM inequality.

9.4 AM-GM-HM inequality

Theorem 9.4.1 *(AM-GM) For $a_1, a_2, ...a_n \geq 0$, we have $\frac{\sum a_i}{n} \geq \sqrt[n]{\prod x_i}$. Equality holds if and only if $a_1 = a_2 = ... = a_n$.*

The proof is omitted.

Example 9.4.2 *Given a,b,c are real number such that $\sigma_1 = 2$, $\sigma_3 = 4$, show that one of them must be positive while the rest is negative.*

By AM-GM inequality, $a + b + c \geq 3\sqrt[3]{abc}$ but it is wrong here. Therefore the three numbers can't be all positive. By checking the sgn of it we can know that two of them are negative.

Example 9.4.3 *Show that $(a+b)(b+c)(c+a) \geq 8abc$ for all positive real.*

Note that when we apply AM-GM inequality, different spliting of the terms lead to a different result, or even weaker than the required result. For example we expand LHS and get $\sum a^2b + \sum b^2a + 2abc$. If we directly apply AM-GM inequality here, we will get

$$(a+b)(b+c)(c+a) \geq 7\sqrt[7]{(\prod(x^2y)(y^2x))(2abc)} = 7\sqrt[7]{2}abc < 8abc$$

.

We can get what we want by spliting $2abc$ into $abc + abc$. Then it becomes

$$(a+b)(b+c)(c+a) \geq 8\sqrt[8]{(\prod(x^2y)(y^2x))(abc)^2} = 8abc$$

.

A quicker alternative method will be $(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8abc$. Which the imequality is applied into each bracket.

Example 9.4.4 *Show that $a^2+b^2+c^2 \geq ab+bc+ca$ for all positive real.*

Now we multiply both sides by 2:

$$(a^2+b^2)+(b^2+c^2)+(c^2+a^2) \geq 2(\sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{c^2a^2}) = 2ab+2bc+2ac$$

.

Note that ab can be bounded by $\frac{1}{2}(a^2+b^2)$ and $(\frac{a+b}{2})^2$.

Example 9.4.5 *Show that $(1+\frac{x}{y})(1+\frac{y}{z})(1+\frac{z}{x}) \geq 2(1+\frac{x+y+z}{\sqrt[3]{xyz}})$. (APMO 1998)*

Obviously we have to solve this by AM-GM inequality.

$$LHS = 2+\sum_{cyc}(\frac{x}{y}+\frac{y}{x}) = 2+\sum_{cyc}(\frac{x}{y}+\frac{x}{z})$$

$$2+\sum_{cyc}(\frac{x}{y}+\frac{x}{z}) = \sum_{cyc}(\frac{x}{y}+\frac{x}{z}+\frac{x}{x})-1 \geq \frac{3(x+y+z)}{\sqrt[3]{xyz}}-1$$

$$\frac{3(x+y+z)}{\sqrt[3]{xyz}}-1 \geq \frac{2(x+y+z)}{\sqrt[3]{xyz}}+\frac{3\sqrt[3]{xyz}}{\sqrt[3]{xyz}}-1 = 2(1+\frac{x+y+z}{\sqrt[3]{xyz}})$$

.

Definition 9.4.6 *The harmonic mean of a set of n numbers is $n(\sum x_i^{-1})^{-1}$.*

Example 9.4.7 *The harmonic mean of 1,2,3 is $3(1+\frac{1}{2}+\frac{1}{3})^{-1} = \frac{18}{11}$.*

Example 9.4.8 *For any positive real $a_1, a_2, ...a_n$, show that if $\prod(1+a_i) = 2^n$, then $\sigma_n \leq 1$.*

Firstly we have to show that $n=1$ is true: if $1+a_1=2$, $a_1=1 \leq 1$.
(Not nesscary but just easy for the case $n=2$ by contradiction: let $a_1a_2 = k > 1$, $\sigma_1 \geq 2\sqrt{\sigma_2} = 2\sqrt{k}$ by AM-GM, then $(1+a_1)(1+a_2) = 1+a_1+a_2+a_1a_2 \geq 1+2\sqrt{k}+k > 4$ which violates our assumption.)
Now assume $n=k$ is true. i.e., $\prod(1+a_i) = 2^n$ and $\sigma_n \leq 1$.
Now we prove by contradiction. Add a positive real a_{n+1} so that $\sigma_{n+1} > 1$. Then $\sigma_{n+1} = (a_1a_2...a_n)(a_{n+1}) > 1$ which implies $a_{n+1} \geq (a_1a_2a_3...a_n)^{-1} \geq 1$.
Now $\prod_{i=1}^{n+1}(1+a_i) = (1+a_{n+1})\prod_{i=1}^{n}(1+a_i) = 2^n(1+a_{n+1}) \geq 2^n(1+1) = 2^{n+1}$ which makes a contradiction.
Therefore the statement must stand. Readers may try to show that the case $k=3$ without using and above result and induction as well.

Example 9.4.9 *Show that* $0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$ *where* x, y, z *are non-negative number and* $x + y + z = 1$. *(IMO 1983)*

For the lower bounding of the inequality, we WLOG let $x \geq y \geq z$, then $z \leq \frac{1}{3}$.

$$xy + yz + zx - 2xyz = xy(1 - 2z) + yz + zx \geq 0$$

For the upper bounding of the inequality, by AM-GM we have $(1-2x)(1-2y)(1-2z) \leq (\frac{1-2x+1-2y+1-2z}{3})^3 = \frac{1}{27}$ when $(1-2x), (1-2y), (1-2z)$ are non-negative. However at most one of them is negative since $x + y + z = 1$, and this is still valid when one of them is negative.

$(1-2x)(1-2x)(1-2z) = 1 - 2(x+y+z) + 4(xy+yz+zx-2xyz) = 4(xy+yz+zx-2xyz) - 1 \leq \frac{1}{27}$

Therefore $xy + yz + zx - 2xyz \leq \frac{7}{27}$.

Now we extend the AM-GM inequality:

Theorem 9.4.10 *The AG-GM-HM inequality tells us that* $n^{-1} \sum x_i \geq (\prod x_i)^{\frac{1}{n}} \geq n(\sum x_i^{-1})^{-1}$ *for all non-negative* x_i. *Equality holds if and only if all* x_i *are equal.*

Example 9.4.11 *Show that* $(a+b+c)(a^{-1}+b^{-1}+c^{-1}) \geq 9$ *for all non-negative real* a, b, c.

We have $\frac{a+b+c}{3} \geq \frac{3}{a^{-1}+b^{-1}+c^{-1}}$ which is true by AM-HM inequality.

Example 9.4.12 *Find the least possible of* $a + \frac{1}{b(a-b)}$ *for* a, b *are postive real.*

We have $a + \frac{1}{b(a-b)} = (a-b) + b + \frac{1}{b(a-b)} \geq 3\sqrt[3]{(a-b)(b)(\frac{1}{b(a-b)})} = 3$, and equality holds when $a = 2$, $b = 1$. Therefore the least possible is 3.

9.5 Cauchy-Schwarz Inequality

Example 9.5.1 *Prove that* $(a^2+c^2)(b^2+d^2) \geq (ab+cd)^2$ *for all non-negative real.*

$$(a^2+c^2)(b^2+d^2) = a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 \geq a^2b^2 + 2abcd + c^2d^2 = (ab+cd)^2$$

The prove is finished by AM-GM inequality. Equality holds if and only if $ad = bc$. As a generalization we will have the Cauchy-Schwarz inequality:

Theorem 9.5.2 *The Cauchy-Schwarz Inequality states that* $(\sum\limits_{i=1}^{n} a_i^2)(\sum\limits_{i=1}^{n} b_i^2) \geq (\sum\limits_{i=1}^{n} a_i b_i)^2$ *for all real numbers. Equality holds if* $(a_1, a_2, ...a_n)$ *can be expressed by* $(rb_1, rb_2, ...rb_n)$, *where* r *is a real number.*

A simplified version of the notation of the inequality will be $(\sum a^2)(\sum b^2) \geq (\sum ab)^2$. Also, we can transform it into $(\sum a_i)(\sum b_i) \geq (\sum \sqrt{ab})^2$ when all of them are non-negative real.

Example 9.5.3 *Show that $a^2+b^2 \geq 2ab$ for all positive real by Cauchy-Schwarz inequality.*

$2(a^2+b^2) = (a^2+b^2)(1+1) \geq (a+b)^2$, by expansion and elimination we will get the required result.

Example 9.5.4 *(9.4.9 revisited) $(a+b+c)(a^{-1}+b^{-1}+c^{-1}) \geq 9$ is obvious by Cauchy-Schwarz inequality for all real.*

Example 9.5.5 *Show that if $x\sqrt{1-y^2}+y\sqrt{1-x^2}=1$, then $x^2+y^2=1$.*

By Cauchy-Schwarz inequality we have

$$x\sqrt{(1-y^2)}+y\sqrt{1-x^2} \leq \sqrt{[x^2+(\sqrt{1-x^2})^2][y^2+(\sqrt{1-y^2})^2]} = 1$$

Equality holds only when $x^2y^2=(1-x^2)(1-y^2)$, i.e., $x^2+y^2=1$.

Example 9.5.6 *Show that $3\sum\limits_{cyc} \frac{1}{a+b+c} \geq \frac{16}{a+b+c+d}$. (AL 2010)*

Example 9.5.7 *Given a, b, c are sides of a triangle and T will be the area. Show that $a^2+b^2+c^2 \geq 4\sqrt{3}T$. (IMO 1961)*

Method I: By cosine law $c^2 = a^2+b^2-2ab\cos C$ and area of triangle $S = \frac{1}{2}ab\sin C$, We will have:

$$\begin{aligned} a^2+b^2+c^2-4\sqrt{3}S &= a^2+b^2+(a^2+b^2-2ab\cos C)-4\sqrt{3}(\frac{1}{2}ab\sin C) \\ &= 2[a^2+b^2-ab(\cos C+\sqrt{3}\sin C)] = 2[a^2+b^2-2ab\sin(C+30^\circ)] \\ &\geq 2(a^2+b^2-2ab) = 2(a-b)^2 \geq 0 \end{aligned}$$

Method II: By herons formula we have

$$T^2 = \frac{1}{4}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = \frac{1}{4}(\sum_{cyc} 2a^2b^2 - a^4)$$

$$\sum a^2 \geq 4\sqrt{3}T = \sqrt{3}\sqrt{(\sum_{cyc} 2a^2b^2 - a^4)}$$

$$\Leftrightarrow (\sum a^2) = \sum(a^4+2a^2b^2) \geq 3(\sum_{cyc} 2a^2b^2 - a^4)$$

$$\Leftrightarrow 4\sum a^4 \geq 4\sum a^2b^2$$

$$\Leftrightarrow 2\sum(a^2-b^2)^2 \geq 0$$

The last line is obviously true since LHS contain squares only.

Example 9.5.8 *Show that* $\sqrt{\frac{\sum x_i^2}{n}} \geq \frac{\sum x_i}{n}$ *for all positive real.*

By squaring both side and time n we have $\sum x_i^2 \geq \frac{1}{n}(\sum x_i)^2$. Now apply Cauchy-Schwarz:

$$\sum x_i^2 = \frac{1}{n}(\sum x_i^2)(\sum 1) \geq \frac{1}{n}(\sum x_i)^2$$

The mean $\sqrt{\frac{\sum x_i^2}{n}}$ is called the square root mean. As a generalization we will have the power mean inequality.

9.6 Rearrangement Inequality

Theorem 9.6.1 *The rearrangement inequality states that for real numbers* $a_1 \geq a_2 \geq a_3 ... \geq a_n$ *and* $b_1 \geq b_2 ... \geq b_n$, *while the set* f_n *are permuatation of* b_n *we will have* $\prod a_i b_i \geq \prod a_1 b_{f_i} \geq \prod a_i b_{n+1-i}$. *Equality holds if and only if* $a_1 = a_2 = ... = a_n$ *OR* $b_1 = b_2 = ... = b_n$.

We can simply say: Direct sum $\geq$ Random sum $\geq$ Reverse sum.

Example 9.6.2 *We have an illustration: let* a_1, a_2, a_3 *be 7,4,2 and* b_1, b_2, b_3 *be 8,6,3 respecitvely.* $(7)(8) + (4)(6) + (2)(3) \geq (7)(6) + (4)(3) + (2)(8) \geq (8)(2) + (4)(6) + (3)(7)$.

Example 9.6.3 *Let* a_i *be a set of positive real numbers and* b_1 *be its permutations. Show that* $\sum \frac{a_n}{b_n} \geq n$.

WLOG we let $a_1 \geq a_2 \geq ... \geq a_n$. Then $\frac{1}{a_1} \leq \frac{1}{a_2} \leq ... \leq \frac{1}{a_n}$. By Random sum $\geq$ Reverse sum we will get the above result.

Example 9.6.4 *Let* x_i *and* y_i *be descending sequence of* n *real number.If* z_i *is a permutation of* y_i, *show that* $\sum(x_i - y_i)^2 \leq \sum(x_i - z_i)^2$. *(IMO 1975)*

Note that we can't apply rearrangement ($x_i - y_i$ VS $x_i - z_i$) directly.

By rearrangement we have $\sum x_i y_i \geq x_i z_i$. Then $\sum(x_i - y_i)^2 - \sum(x_i - z_i)^2 = 2\sum x_i(z_i - y_i) \geq 0$.

Note that $\sum y_i^2 = \sum z_i^2$.

Example 9.6.5 *Show that* $a^2 + b^2 + c^2 \geq ab + bc + ca$ *for positive real.*

WLOG let $a \geq b \geq c$. By direct sum $\geq$ random sum (We treat it as random since b, c, a doesn't increase/decrease monotonically in order.) we get the above claim.

Example 9.6.6 *Show that* $\sum\limits_{cyc} \frac{a}{b+c} \geq \frac{3}{2}$.

Since we can't decomose the denominator easily, we may expand it by multiply both sides by $(a+b)(b+c)(c+a)$.

$$2\sum a(a+b)(a+c) \geq 3(a+b)(b+c)(c+a)$$

$$2(a^3+b^3+c^3+3abc+\sum(a^2b+b^2a)) \geq 3(\sum(a^2b+b^2a)+2abc)$$

$$2(a^3+b^3+c^3) \geq \sum(a^2b+b^2a)$$

Which is easily proven by rearrangement inequality.

When there are more than two variables we may have to use rearrangement inequality for twice or more.

Example 9.6.7 *Show that $a^3+b^3+c^3 \geq 3abc$ for all positive real by rearrangement inequality.*

WLOG we let $a \geq b \geq c$, then we have $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$ by direct sum $\geq$ random sum (a^2, b^2, c^2 VS a, b, c) and $a^2b+b^2c+c^2a \geq 3abc$ by direct sum $\geq$ random sum (ab, ac, bc VS a, b, c).

Example 9.6.8 *Show that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{a^8+b^8+c^8}{a^3b^3c^3}$ for all positive real.*

We have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+bc+ca}{abc} \leq \frac{a^8+b^8+c^8}{a^3b^3c^3}$

$$a^2b^2c^2(ab+bc+ca) = a^3b^3c^2+a^3b^2c^3+a^2b^3c^3 \leq a^8+b^8+c^8$$

By rearrangement we have $a^8+b^8+c^8 \geq a^2b^6+b^2c^6+c^2a^6 \geq a^2(b^3c^3)+b^2(c^3a^3)+c^2(a^3b^3)$.

Note that we apply rearrangement inequality for the second time since random sum $\geq$ reverse sum since $(bc)^3 \leq (ac)^3 \leq (ab)^3$.

When we apply rearrangement inequality, we should aware that the dummy terms (a_i) should be in decending order. Otherwise we can't compare if both a_i and b_i are in random order. e.g. We can't say $a^6c^2+b^6a^2+c^2b^2 \geq a^2(b^3c^3)+b^2(c^3a^3)+c^2(a^3b^3)$.

9.7 Other famous inequalities

Theorem 9.7.1 *The Power Mean Inequality states that $(\frac{\sum x_i^p}{n})^{\frac{1}{p}} \geq (\frac{\sum x_i^s}{n})^{\frac{1}{s}}$ as long as $p \geq s$, where p, s are real number and x_i are positive real.*

Definition 9.7.2 *For a set of positive real number X, where $|x| = n$, we denote $(\frac{\sum x_i^p}{n})^{\frac{1}{p}}$ as M_p where $p \leq n$..*

Now let's observe some special cases:

Corollary 9.7.3 $\lim_{p\to\infty} M_p = maxx_1, x_2, ...x_n$. $\lim_{p\to-\infty} M_p = minx_1, x_2, ...x_n$

Proof: Firstly we know that when p is large enough, $(x+1)^p - x^p$ is a large number, so when p is getting larger, the difference between $x_1, x_2, ...x_n$ will be monotonically increase. When p tends to infinity, the largest term actually held the mean of it. Therefore we get the above corollary.

Corollary 9.7.4 *M_2 Is the square root mean of the number set while M_1 is the arithmetic mean of the numbers.*

By the Power Mean Inequality we now know that why the above example is correct.

Corollary 9.7.5 *$\lim_{p\to 0} M_p$ is the geometric mean of the number set.*

The proof is omitted.

Definition 9.7.6 *Denote Let X be a set of k postive real numbers. For $k \geq n$ we have P_n as $\frac{\sigma_n}{C_n^k} = \frac{n!(k-n)!\sigma_n}{k!}$. It represents the average of all of the products of n a_i's.*

The following will be another generalization of the AM-GM inequality since P_1 is the AM while $(P_n)^{\frac{1}{n}}$ is the GM of the number.

Theorem 9.7.7 *The Maclaurin's Symmetric Mean Inequality states that $P_1 \geq P_2^{\frac{1}{2}} \geq ... \geq P_n^{\frac{1}{n}}$.*

Example 9.7.8 *Show that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8+b^8+c^8}{a^3b^3c^3}$.*

We have to use Power Mean inequality and Maclaurin's Symmetric Mean Inequality together to solve this inequality.

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} = \frac{a^8+b^8+c^8}{3}(\frac{3}{\sigma_3^3}) = M_8^8(\frac{3}{\sigma_3^3})$$

$$M_8^8(\frac{3}{\sigma_3^3}) \geq M_1^8(\frac{3}{\sigma_3^3}) = P_1^8(\frac{3}{\sigma_3^3}) = (P_1^6)(P_1^2)(\frac{3}{\sigma_3^3})$$

$$(P_1^6)(P_1^2)(\frac{3}{\sigma_3^3}) \geq (P_3^{\frac{1}{3}})^6(P_2^{\frac{1}{2}})^2(\frac{3}{\sigma_3^3}) = (\frac{\sigma_3^2\sigma_2}{3})(\frac{3}{\sigma_3^3}) = \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Note that $M_1 = P_1$ which is AM.

Theorem 9.7.9 *Jensen's inequality states that for any convex function, $\sum a_i f(x_i) \leq f(\sum a_i x_i)$ where $\sum a_i = 1$. If the functino is concave, then $\sum a_i f(x_i) \geq f(\sum a_i x_i)$.*

Example 9.7.10 *$f(x) = \frac{1}{\sqrt{x}}$ is a concave function.*

Example 9.7.11 *Show that $\sum\limits_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq 1$ for all positive real. (IMO 2001)*

The solution will be a bit complicated. Firstly we will discuss the solution involving Jensen's inequality.

Let $f(x) = \frac{1}{\sqrt{x}}$ and it is concave.

Now $\sum\limits_{cyc} \frac{af(a^2+8ac)}{a+b+c} \geq f(\frac{a^3+b^3+c^3+24abc}{a+b+c}) = \frac{\sqrt{a+b+c}}{\sqrt{a^3+b^3+c^3+24abc}}$, multiply both sides by $a+b+c$ gives $\sum\limits_{cyc} \frac{a}{\sqrt{a^2+8bc}} \geq \frac{\sqrt{(a+b+c)^3}}{\sqrt{a^3+b^3+c^3+24abc}} \geq 1$ since $(a+b+c)^3 \geq a^3+b^3+c^3+24abc$ by AM-GM inequality.

9.8 Problems

1. Show that $\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + ... + \frac{a_n^2}{a_1} \geq \sum a_i$ for all positive real.
2. Show that $H_n \leq \sum \frac{a_i}{i^2}$ where H_n are the partial sum of harmonic series and a_i are distinct positive integers.
3. Show that $\sum\limits_{cyc} \frac{1}{a^3(b+c)} \geq \frac{3}{2}$ where a, b, c are positive real with $abc = 1$. (IMO 1995)
4. Let a, b, c are sides of a triangle. Show that $\sum\limits_{cyC} a^2b(a-b) \geq 0$. (IMO 1983)
5. Show that $\frac{3}{5} \leq \sum\limits_{cyc} \frac{a}{a+2b+2c} > 1$ where a, b, c are positive real.
6. Show that $(a+b)^2 + (a+b+4c)^2 \leq \frac{100abc}{a+b+c}$ where a, b, c are positive real.
7. Show that $\prod\limits_{cyc}(x+y-z) \leq xyz$ for all positive real.
8. Show that $(x^2+y^2+z^2)^3 \geq 54x^2y^2z^2$ for all positive real with $z = x+y$.
9. Show that $\sigma_2 - 3\sigma_3 \leq \frac{1}{4}$ for all positive real with $\sigma_1 = 1$.
10. Show that $\sum\limits_{cyc} \frac{1}{a^3+b^3+abc} \leq \frac{1}{abc}$ for all positive real. (USAMO 1997).
11. Show that $\sum\limits_{cyc} \frac{x^5-x^2}{x^5+y^2+z^2} \geq 0$ for all positive real with $xyz \geq 1$. (IMO 2005)

Chapter 10

Function(III)

10.1 Properties of functions

Definition 10.1.1 *We define a function as a rule of mapping every value in the domain to exactly one value in the co-domain.*

Chapter 11

Appendix

11.1 Frequently used notation

Set notation

- $a \in X$ implies a is in the set X. $a \notin X$ implies a isn't in the set X.
- We say $X = \{a_1, a_2, ...a_n\}$, where a_i will be its elements.
- We denote the set of positive integers, integers, rational numbers, real numbers and complex numbers by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively.
- We denote $\mathbb{Z}_0, \mathbb{R}_0$ for non-zero integers and real numbers respectively.
- We denote $\mathbb{Z}^+, \mathbb{R}^+$ for positive intergers and positive real respectively.
- $\exists k$ means there exist a k to satisfy the previous assumption.
- $\forall X$ means for all elements in the set X, it satisfy the previous assumption.

Cyclic notation

- The elementary function σ_k is the sum of any combination of the product of k elements from the n elements given from the sets.
- The cyclic sum is defined as

$$\sum_{cyc} f(a_1, a_2, ...a_n) = f(a_1, a_2, ...a_n)+f(a_2, a_3, ...a_n, a_1)+...+f(a_n, a_1, ...a_{n-1})$$

.

- The cyclic product is defined as

$$\prod_{cyc} f = f(a_1, a_2, ...a_n)f(a_2, a_3, ...a_n, a_1)...f(a_n, a_1, ...a_{n-1})$$

.