CONCEPT CHECK

1. (a) See Definition 11.1.1.

   (b) See Definition 11.2.2.

   (c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

   (d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 11.1.11.

   (b) A sequence is monotonic if it is either increasing or decreasing.

   (c) By Theorem 11.1.12, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 11.2.

   (b) The p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent if $p > 1$.

4. If $\sum a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. (a) *Test for Divergence:* If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

   (b) *Integral Test:* Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:
   (i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
   (ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

   (c) *Comparison Test:* Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
   (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
   (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

   (d) *Limit Comparison Test:* Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim\limits_{n\to\infty} (a_n/b_n) = c$, where c is a finite number and $c > 0$, then either both series converge or both diverge.

(e) *Alternating Series Test:* If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad [b_n > 0]$ satisfies (i) $b_{n+1} \leq b_n$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

(f) *Ratio Test:*

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

(g) *Root Test:*

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive..

6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

(b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

(c) A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

7. (a) Use (3) in Section 11.3.

(b) See Example 5 in Section 11.4.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 712.

8. (a) $\sum_{n=0}^{\infty} c_n(x-a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 11.9.2.

10. (a) $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ [$a = 0$ in part (b)]

(d) See Theorem 11.10.8.

(e) See Taylor's Inequality (11.10.9).

11. (a)–(e) See Table 1 on page 743.

12. See the binomial series (11.10.17) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 11.2.6.

2. False. The series $\sum\limits_{n=1}^{\infty} n^{-\sin 1} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\sin 1}}$ is a p-series with $p = \sin 1 \approx 0.84 \le 1$, so the series diverges.

3. True. If $\lim\limits_{n\to\infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$.

If $n > N$, then $2n + 1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim\limits_{n\to\infty} a_{2n+1} = L$.

4. True by Theorem 11.8.3.

Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.

5. False. For example, take $c_n = (-1)^n/(n6^n)$.

6. True by Theorem 11.8.3.

7. False, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{1}{(n+1)^3} \cdot \dfrac{n^3}{1}\right| = \lim\limits_{n\to\infty} \left|\dfrac{n^3}{(n+1)^3} \cdot \dfrac{1/n^3}{1/n^3}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1$.

8. True, since $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{1}{(n+1)!} \cdot \dfrac{n!}{1}\right| = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 < 1$.

9. False. See the note after Example 2 in Section 11.4.

10. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n!}$.

11. True. See (9) in Section 11.1.

12. True, because if $\sum |a_n|$ is convergent, then so is $\sum a_n$ by Theorem 11.6.3.

13. True. By Theorem 11.10.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \Rightarrow f'''(0) = 2$.

Or: Use Theorem 11.9.2 to differentiate f three times.

14. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \Rightarrow \{a_n\}$ is bounded.

17. True by Theorem 11.6.3. [$\sum (-1)^n a_n$ is absolutely convergent and hence convergent.]

18. True. $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n$ converges (Ratio Test) $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$ [Theorem 11.2.6].

19. True. $0.99999\ldots = 0.9 + 0.9(0.1)^1 + 0.9(0.1)^2 + 0.9(0.1)^3 + \cdots = \sum\limits_{n=1}^{\infty} (0.9)(0.1)^{n-1} = \dfrac{0.9}{1-0.1} = 1$ by the formula for the sum of a geometric series [$S = a_1/(1-r)$] with ratio r satisfying $|r| < 1$.

20. False. Let $a_n = (0.1)^n$ and $b_n = (0.2)^n$. Then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (0.1)^n = \dfrac{0.1}{1-0.1} = \dfrac{1}{9} = A$,

$\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} (0.2)^n = \dfrac{0.2}{1-0.2} = \dfrac{1}{4} = B$, and $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} (0.02)^n = \dfrac{0.02}{1-0.02} = \dfrac{1}{49}$, but

$AB = \frac{1}{9} \cdot \frac{1}{4} = \frac{1}{36}$.

EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\lim\limits_{n\to\infty} \dfrac{2+n^3}{1+2n^3} = \lim\limits_{n\to\infty} \dfrac{2/n^3+1}{1/n^3+2} = \dfrac{1}{2}$.

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9 \lim\limits_{n\to\infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$ by (11.1.9).

3. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^3}{1+n^2} = \lim\limits_{n\to\infty} \dfrac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $a_n = \cos(n\pi/2)$, so $a_n = 0$ if n is odd and $a_n = \pm 1$ if n is even. As n increases, a_n keeps cycling through the values 0, 1, 0, -1, so the sequence $\{a_n\}$ is divergent.

5. $|a_n| = \left|\dfrac{n \sin n}{n^2+1}\right| \le \dfrac{n}{n^2+1} < \dfrac{1}{n}$, so $|a_n| \to 0$ as $n \to \infty$. Thus, $\lim\limits_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ for $x > 0$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1/(2\sqrt{x})} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$.
Thus, by Theorem 3 in Section 11.1, $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n = 0$.

7. $\left\{\left(1+\dfrac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y = \left(1+\dfrac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 4x \ln(1+3/x) = \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/(4x)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+3/x}\left(-\dfrac{3}{x^2}\right)}{-1/(4x^2)} = \lim_{x\to\infty} \frac{12}{1+3/x} = 12, \text{ so}$$

$$\lim_{x\to\infty} y = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{4n} = e^{12}.$$

8. $\left\{\dfrac{(-10)^n}{n!}\right\}$ converges, since $\dfrac{10^n}{n!} = \dfrac{10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot 10} \cdot \dfrac{10 \cdot 10 \cdot \cdots \cdot 10}{11 \cdot 12 \cdot \cdots \cdot n} \le 10^{10} \left(\dfrac{10}{11}\right)^{n-10} \to 0$ as $n \to \infty$, so

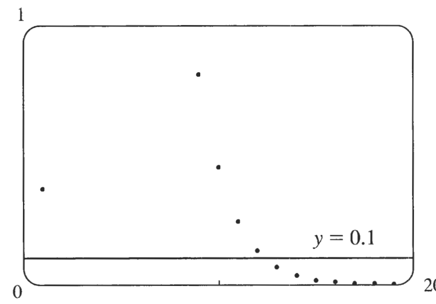
$\lim\limits_{n\to\infty} \dfrac{(-10)^n}{n!} = 0$ [Squeeze Theorem]. *Or:* Use (11.10.10).

9. We use induction, hypothesizing that $a_{n-1} < a_n < 2$. Note first that $1 < a_2 = \frac{1}{3}(1+4) = \frac{5}{3} < 2$, so the hypothesis holds for $n = 2$. Now assume that $a_{k-1} < a_k < 2$. Then $a_k = \frac{1}{3}(a_{k-1}+4) < \frac{1}{3}(a_k+4) < \frac{1}{3}(2+4) = 2$. So $a_k < a_{k+1} < 2$, and the induction is complete. To find the limit of the sequence, we note that $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \Rightarrow$ $L = \frac{1}{3}(L+4) \Rightarrow L = 2$.

10. $\lim\limits_{x\to\infty} \dfrac{x^4}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{4x^3}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{12x^2}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{24}{e^x} = 0$

Then we conclude from Theorem 11.1.3 that $\lim\limits_{n\to\infty} n^4 e^{-n} = 0$.

From the graph, it seems that $12^4 e^{-12} > 0.1$, but $n^4 e^{-n} < 0.1$ whenever $n > 12$. So the smallest value of N corresponding to $\varepsilon = 0.1$ in the definition of the limit is $N = 12$.



11. $\dfrac{n}{n^3+1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ $[p = 2 > 1]$.

12. Let $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, so $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3+1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$.

Since $\sum\limits_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum\limits_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

13. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^3}{5^{n+1}} \cdot \dfrac{5^n}{n^3}\right] = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^3 \cdot \dfrac{1}{5} = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n = \dfrac{1}{\sqrt{n+1}}$. Then b_n is positive for $n \geq 1$, the sequence $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\begin{aligned}\int_2^\infty f(x)\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \left[u = \ln x,\, du = \tfrac{1}{x}\,dx\right] = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-1/2}\,du = \lim_{t\to\infty} \left[2\sqrt{u}\right]_{\ln 2}^{\ln t} \\ &= \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty,\end{aligned}$$

so the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

16. $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{3n+1}\right) = \ln\frac{1}{3} \neq 0$. Thus, the series $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $|a_n| = \left|\dfrac{\cos 3n}{1+(1.2)^n}\right| \leq \dfrac{1}{1+(1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6}\right)^n$, so $\sum\limits_{n=1}^{\infty} |a_n|$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ $\left[r = \frac{5}{6} < 1\right]$. It follows that $\sum\limits_{n=1}^{\infty} a_n$ converges (by Theorem 3 in Section 11.6).

18. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{2n}}{(1+2n^2)^n}\right|} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \lim_{n\to\infty} \frac{1}{1/n^2+2} = \frac{1}{2} < 1$, so $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ converges by the Root Test.

19. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!} \cdot \frac{5^n\, n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \lim_{n\to\infty} \frac{2n+1}{5(n+1)} = \frac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2\, 9^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{25}{9}\right)^n$. Now $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{25^{n+1}}{(n+1)^2\cdot 9^{n+1}} \cdot \frac{n^2\cdot 9^n}{25^n} = \lim_{n\to\infty} \frac{25n^2}{9(n+1)^2} = \frac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. $b_n = \frac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n = \frac{1}{n^{3/2}}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum_{n=1}^{\infty} b_n$ converges $\left[p = \frac{3}{2} > 1\right]$, $\sum_{n=1}^{\infty} a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^{\infty} n^{-1/3}$ is a p-series with $p = \frac{1}{3} \le 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \frac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1}\, n^{-1/3}$ converges. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1}\, n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty} \left|(-1)^{n-1}\, n^{-3}\right| = \sum_{n=1}^{\infty} n^{-3}$ is a convergent p-series $[p = 3 > 1]$. Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1}\, n^{-3}$ is absolutely convergent.

25. $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+2)3^{n+1}}{2^{2n+3}} \cdot \frac{2^{2n+1}}{(-1)^n(n+1)3^n}\right| = \frac{n+2}{n+1}\cdot\frac{3}{4} = \frac{1+(2/n)}{1+(1/n)}\cdot\frac{3}{4} \to \frac{3}{4} < 1$ as $n\to\infty$, so by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{1/x} = \lim_{x\to\infty} \frac{\sqrt{x}}{2} = \infty$. Therefore, $\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{\ln n} \neq 0$, so the given series is divergent by the Test for Divergence.

27. $$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{(2^3)^n} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^n} = \frac{1}{8}\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{8^{n-1}} = \frac{1}{8}\sum_{n=1}^{\infty}\left(-\frac{3}{8}\right)^{n-1} = \frac{1}{8}\left(\frac{1}{1-(-3/8)}\right)$$
$$= \frac{1}{8}\cdot\frac{8}{11} = \frac{1}{11}$$

28. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)}=\sum_{n=1}^{\infty}\left[\frac{1}{3n}-\frac{1}{3(n+3)}\right]$ [partial fractions].

$\displaystyle s_n=\sum_{i=1}^{n}\left[\frac{1}{3i}-\frac{1}{3(i+3)}\right]=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}-\frac{1}{3(n+1)}-\frac{1}{3(n+2)}-\frac{1}{3(n+3)}$ (telescoping sum), so

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)}=\lim_{n\to\infty}s_n=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}=\frac{11}{18}.$

29. $\displaystyle\sum_{n=1}^{\infty}[\tan^{-1}(n+1)-\tan^{-1}n]=\lim_{n\to\infty}s_n$

$$=\lim_{n\to\infty}[(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+\cdots+(\tan^{-1}(n+1)-\tan^{-1}n)]$$

$$=\lim_{n\to\infty}[\tan^{-1}(n+1)-\tan^{-1}1]=\tfrac{\pi}{2}-\tfrac{\pi}{4}=\tfrac{\pi}{4}$$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\frac{\pi^n}{3^{2n}}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\left(\frac{\sqrt{\pi}}{3}\right)^{2n}=\cos\left(\frac{\sqrt{\pi}}{3}\right)$ since $\displaystyle\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ for all x.

31. $\displaystyle 1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{e^n}{n!}=\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}=e^{-e}$ since $\displaystyle e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

32. $\displaystyle 4.17\overline{326}=4.17+\frac{326}{10^5}+\frac{326}{10^8}+\cdots=4.17+\frac{326/10^5}{1-1/10^3}=\frac{417}{100}+\frac{326}{99{,}900}=\frac{416{,}909}{99{,}900}$

33. $\displaystyle\cosh x=\frac{1}{2}(e^x+e^{-x})=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)$

$$=\frac{1}{2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots\right)\right]$$

$$=\frac{1}{2}\left(2+2\cdot\frac{x^2}{2!}+2\cdot\frac{x^4}{4!}+\cdots\right)=1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!}\geq 1+\frac{1}{2}x^2\quad\text{for all } x$$

34. $\displaystyle\sum_{n=1}^{\infty}(\ln x)^n$ is a geometric series which converges whenever $|\ln x|<1\ \Rightarrow\ -1<\ln x<1\ \Rightarrow\ e^{-1}<x<e.$

35. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}=1-\frac{1}{32}+\frac{1}{243}-\frac{1}{1024}+\frac{1}{3125}-\frac{1}{7776}+\frac{1}{16{,}807}-\frac{1}{32{,}768}+\cdots.$

Since $\displaystyle b_8=\frac{1}{8^5}=\frac{1}{32{,}768}<0.000031,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\approx\sum_{n=1}^{7}\frac{(-1)^{n+1}}{n^5}\approx 0.9721.$

36. (a) $\displaystyle s_5=\sum_{n=1}^{5}\frac{1}{n^6}=1+\frac{1}{2^6}+\cdots+\frac{1}{5^6}\approx 1.017305.$ The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (11.3.2): $\displaystyle R_5\leq\int_5^{\infty}\frac{dx}{x^6}=\left[-\frac{x^{-5}}{5}\right]_5^{\infty}=\frac{5^{-5}}{5}=0.000064.$ So the error is at most 0.000064.

(b) In general, $\displaystyle R_n\leq\int_n^{\infty}\frac{dx}{x^6}=\frac{1}{5n^5}.$ If we take $n=9$, then $s_9\approx 1.01734$ and $\displaystyle R_9\leq\frac{1}{5\cdot 9^5}\approx 3.4\times 10^{-6}.$

So to five decimal places, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5}\approx\sum_{n=1}^{9}\frac{1}{n^5}\approx 1.01734.$

Another method: Use (11.3.3) instead of (11.3.2).

37. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2+5^n} \approx \sum\limits_{n=1}^{8} \dfrac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\dfrac{1}{2+5^n} < \dfrac{1}{5^n}$, so the remainder term is

$R_8 = \sum\limits_{n=9}^{\infty} \dfrac{1}{2+5^n} < \sum\limits_{n=9}^{\infty} \dfrac{1}{5^n} = \dfrac{1/5^9}{1-1/5} = 6.4 \times 10^{-7}$ $\left[\text{geometric series with } a = \frac{1}{5^9} \text{ and } r = \frac{1}{5}\right]$.

38. (a) $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)^{n+1}}{[2(n+1)]!} \cdot \dfrac{(2n)!}{n^n}\right| = \lim\limits_{n\to\infty} \dfrac{(n+1)^n(n+1)^1}{(2n+2)(2n+1)n^n} = \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n \dfrac{1}{2(2n+1)}$

$= \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n \dfrac{1}{2(2n+1)} = e \cdot 0 = 0 < 1$

so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^n}{(2n)!} = 0$ by Theorem 11.2.6.

39. Use the Limit Comparison Test. $\lim\limits_{n\to\infty} \left|\dfrac{\left(\frac{n+1}{n}\right)a_n}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{n+1}{n} = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right) = 1 > 0$.

Since $\sum |a_n|$ is convergent, so is $\sum \left|\left(\dfrac{n+1}{n}\right)a_n\right|$, by the Limit Comparison Test.

40. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)^2\, 5^{n+1}} \cdot \dfrac{n^2 5^n}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^2} \dfrac{|x|}{5} = \dfrac{|x|}{5}$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^n}{n^2\, 5^n}$

converges when $\dfrac{|x|}{5} < 1 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$. When $x = -5$, the series becomes the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ with

$p = 2 > 1$. When $x = 5$, the series becomes $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I = [-5, 5]$.

41. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left[\dfrac{|x+2|^{n+1}}{(n+1)\, 4^{n+1}} \cdot \dfrac{n\, 4^n}{|x+2|^n}\right] = \lim\limits_{n\to\infty} \left[\dfrac{n}{n+1} \dfrac{|x+2|}{4}\right] = \dfrac{|x+2|}{4} < 1 \quad \Leftrightarrow \quad |x+2| < 4$, so $R = 4$.

$|x+2| < 4 \quad \Leftrightarrow \quad -4 < x+2 < 4 \quad \Leftrightarrow \quad -6 < x < 2$. If $x = -6$, then the series $\sum\limits_{n=1}^{\infty} \dfrac{(x+2)^n}{n\, 4^n}$ becomes

$\sum\limits_{n=1}^{\infty} \dfrac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x = 2$, the

series becomes the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

42. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2^{n+1}\,(x-2)^{n+1}}{(n+3)!} \cdot \dfrac{(n+2)!}{2^n (x-2)^n}\right| = \lim\limits_{n\to\infty} \dfrac{2}{n+3} |x-2| = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n\,(x-2)^n}{(n+2)!}$

converges for all x. $R = \infty$ and $I = (-\infty, \infty)$.

43. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}} \cdot \dfrac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2\,|x-3| \lim\limits_{n\to\infty} \sqrt{\dfrac{n+3}{n+4}} = 2\,|x-3| < 1 \quad \Leftrightarrow \quad |x-3| < \frac{1}{2}$,

so $R = \frac{1}{2}$. $|x-3| < \frac{1}{2} \quad \Leftrightarrow \quad -\frac{1}{2} < x-3 < \frac{1}{2} \quad \Leftrightarrow \quad \frac{5}{2} < x < \frac{7}{2}$. For $x = \frac{7}{2}$, the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes

$\sum\limits_{n=0}^{\infty} \dfrac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty} \dfrac{1}{n^{1/2}}$, which diverges $\left[p = \frac{1}{2} \le 1\right]$, but for $x = \frac{5}{2}$, we get $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{\sqrt{n+3}}$, which is a convergent

alternating series, so $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

44. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(2n+2)!\,x^{n+1}}{[(n+1)!]^2}\cdot\dfrac{(n!)^2}{(2n)!\,x^n}\right| = \lim\limits_{n\to\infty}\dfrac{(2n+2)(2n+1)}{(n+1)(n+1)}\,|x| = 4\,|x|.$

To converge, we must have $4\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

45.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\sin x &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots \\
&= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}
\end{aligned}$$

46.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
\cos x &= f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots \\
&= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[-\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 - \cdots\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}
\end{aligned}$$

47. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1 \quad\Rightarrow\quad \dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R = 1$.

48. $\tan^{-1} x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1, 1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2 \in [-1, 1] \quad\Leftrightarrow\quad x \in [-1, 1]$.

Therefore, $R = 1$.

49. $\dfrac{1}{1-x}=\sum\limits_{n=0}^{\infty}x^n$ for $|x|<1 \;\Rightarrow\; \ln(1-x)=-\displaystyle\int\frac{dx}{1-x}=-\int\sum_{n=0}^{\infty}x^n\,dx=C-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$.

$\ln(1-0)=C-0 \;\Rightarrow\; C=0 \;\Rightarrow\; \ln(1-x)=-\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{-x^n}{n}$ with $R=1$.

50. $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \;\Rightarrow\; e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} \;\Rightarrow\; xe^{2x}=x\sum_{n=0}^{\infty}\frac{2^n\,x^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n\,x^{n+1}}{n!},\; R=\infty$

51. $\sin x=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2n+1)!} \;\Rightarrow\; \sin(x^4)=\sum_{n=0}^{\infty}\frac{(-1)^n\,(x^4)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

52. $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \;\Rightarrow\; 10^x=e^{(\ln 10)x}=\sum_{n=0}^{\infty}\frac{[(\ln 10)x]^n}{n!}=\sum_{n=0}^{\infty}\frac{(\ln 10)^n x^n}{n!},\; R=\infty$

53.
$$f(x)=\frac{1}{\sqrt[4]{16-x}}=\frac{1}{\sqrt[4]{16(1-x/16)}}=\frac{1}{\sqrt[4]{16}\left(1-\frac{1}{16}x\right)^{1/4}}=\tfrac{1}{2}\left(1-\tfrac{1}{16}x\right)^{-1/4}$$
$$=\frac{1}{2}\left[1+\left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right)+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\frac{x}{16}\right)^3+\cdots\right]$$
$$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot5\cdot9\cdot\cdots\cdot(4n-3)}{2\cdot4^n\cdot n!\cdot16^n}\,x^n=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot5\cdot9\cdot\cdots\cdot(4n-3)}{2^{6n+1}\,n!}\,x^n$$
for $\left|-\dfrac{x}{16}\right|<1 \;\Leftrightarrow\; |x|<16$, so $R=16$.

54.
$$(1-3x)^{-5}=\sum_{n=0}^{\infty}\binom{-5}{n}(-3x)^n=1+(-5)(-3x)+\frac{(-5)(-6)}{2!}(-3x)^2+\frac{(-5)(-6)(-7)}{3!}(-3x)^3+\cdots$$
$$=1+\sum_{n=1}^{\infty}\frac{5\cdot6\cdot7\cdot\cdots\cdot(n+4)\cdot3^n\,x^n}{n!}$$ for $|-3x|<1 \;\Leftrightarrow\; |x|<\frac{1}{3}$, so $R=\frac{1}{3}$.

55. $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, so $\dfrac{e^x}{x}=\dfrac{1}{x}\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{n-1}}{n!}=x^{-1}+\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$ and

$\displaystyle\int\frac{e^x}{x}\,dx=C+\ln|x|+\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$.

56.
$$(1+x^4)^{1/2}=\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(x^4)^n=1+\left(\tfrac{1}{2}\right)x^4+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3+\cdots$$
$$=1+\tfrac{1}{2}x^4-\tfrac{1}{8}x^8+\tfrac{1}{16}x^{12}-\cdots$$
so $\int_0^1(1+x^4)^{1/2}\,dx=\left[x+\frac{1}{10}x^5-\frac{1}{72}x^9+\frac{1}{208}x^{13}-\cdots\right]_0^1=1+\frac{1}{10}-\frac{1}{72}+\frac{1}{208}-\cdots$.

This is an alternating series, so by the Alternating Series Test, the error in the approximation $\int_0^1(1+x^4)^{1/2}\,dx\approx1+\frac{1}{10}-\frac{1}{72}\approx1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.

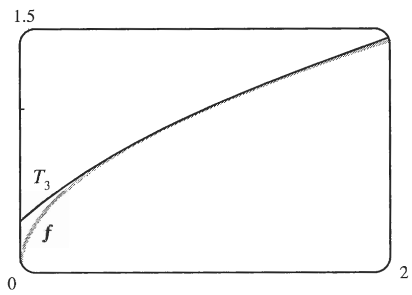
Thus, correct to two decimal places, $\int_0^1(1+x^4)^{1/2}\,dx\approx1.09$.

57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\sqrt{x} \approx T_3(x) &= 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3\end{aligned}$$

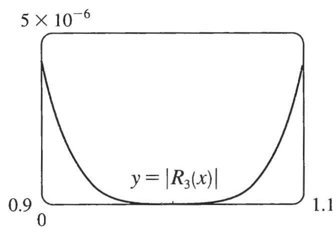
(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \Rightarrow -0.1 \le x - 1 \le 0.1 \Rightarrow (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \le \frac{15}{16(0.9)^{7/2}\,4!}(0.1)^4 \approx 0.000\,005\,648$$
$$\approx 0.000\,006 = 6 \times 10^{-6}$$

(d)



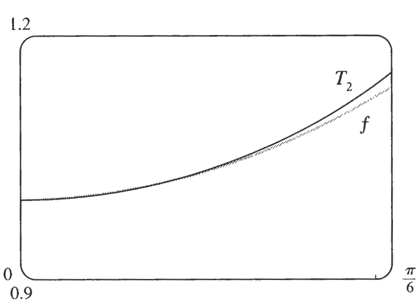
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

58. (a)

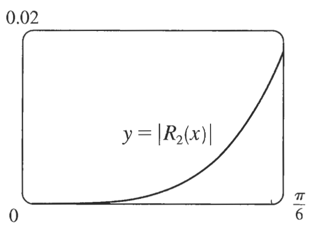
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5\sec^3 x \tan x$	0
$\vdots$	$\vdots$	$\vdots$

$$\sec x \approx T_2(x) = 1 + \tfrac{1}{2}x^2$$

(b)



(c) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$ with $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^3 \le \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so $|R_2(x)| \le \dfrac{14}{3 \cdot 6}\left(\dfrac{\pi}{6}\right)^3 \approx 0.111648$.

(d) 

From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

59. $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, so $\sin x - x = -\dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ and

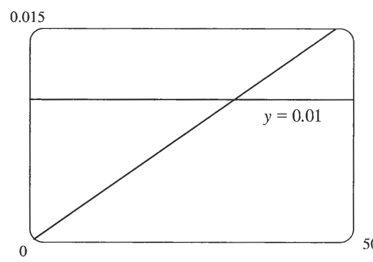
$\dfrac{\sin x - x}{x^3} = -\dfrac{1}{3!} + \dfrac{x^2}{5!} - \dfrac{x^4}{7!} + \cdots$. Thus, $\lim_{x \to 0} \dfrac{\sin x - x}{x^3} = \lim_{x \to 0} \left(-\dfrac{1}{6} + \dfrac{x^2}{120} - \dfrac{x^4}{5040} + \cdots \right) = -\dfrac{1}{6}$.

60. (a) $F = \dfrac{mgR^2}{(R+h)^2} = \dfrac{mg}{(1+h/R)^2} = mg \sum_{n=0}^{\infty} \binom{-2}{n} \left(\dfrac{h}{R}\right)^n$ [binomial series]

(b) We expand $F = mg\left[1 - 2\,(h/R) + 3\,(h/R)^2 - \cdots\right]$.

This is an alternating series, so by the Alternating Series Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want

$\left| \dfrac{2mgh/R}{mgR^2/(R+h)^2} \right| < 0.01 \quad \Leftrightarrow \quad \dfrac{2h(R+h)^2}{R^3} < 0.01.$



This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.

61. $f(x) = \sum_{n=0}^{\infty} c_n\, x^n \quad \Rightarrow \quad f(-x) = \sum_{n=0}^{\infty} c_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n c_n\, x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 11.10.5), so $(-1)^n\, c_n = -c_n$.

If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0, that is, $c_0 = c_2 = c_4 = \cdots = 0$.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n\, c_n\, x^n = \sum_{n=0}^{\infty} c_n\, x^n \quad \Rightarrow \quad (-1)^n\, c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \cdots = 0$.

62. $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \Rightarrow \quad f(x) = e^{x^2} = \sum_{n=0}^{\infty} \dfrac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \dfrac{1}{n!} x^{2n}$. By Theorem 11.10.6 with $a = 0$, we also have

$f(x) = \sum_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!} x^k$. Comparing coefficients for $k = 2n$, we have $\dfrac{f^{(2n)}(0)}{(2n)!} = \dfrac{1}{n!} \quad \Rightarrow \quad f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.