Problem Set 1

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1. (a) Under translation of coordinates, the components of a vector do not change.
   (b) Under inversion of coordinates, the components of a vector are multiplied by -1.
   (c) Under inversion of coordinates, $\mathbf{C} = \mathbf{A} \times \mathbf{B} \to (-\mathbf{A}) \times (-\mathbf{B}) = \mathbf{C}$, that is, its components do not change. The cross product of two pseudovectors is itself a pseudovector. Two examples of pseudovectors are angular momentum and torque.
   (d) We have $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \to (-\mathbf{A}) \cdot ((-\mathbf{B}) \times (-\mathbf{C})) = -\mathbf{A}\dot{(}\mathbf{B} \times \mathbf{C})$, so the scalar triple product changes sign under inversion of coordinates.
2. (a) $h(x, y)$ is maximized at $x = 0$, $y = -1/5$, so the top of the hill is 0.2 miles or 321.869 meters south of the VPC.
   (b) $h(0, -1/5) = 444$ feet high.
   (c) If we solve $h(x, -1/5) = 23$, we obtain $x = 0.58$. The gradient is given by

   $$\nabla h(x, y) = \hat{\mathbf{x}} \frac{\partial h}{\partial x} + \hat{\mathbf{y}} \frac{\partial h}{\partial y} = -2500x\,\hat{\mathbf{x}} - 180(5y + 1)\,\hat{\mathbf{y}}$$

   Its value at $(0.58, -0.2)$ is $-1450\,\hat{\mathbf{x}}$. Thus, the steepness of the slope is 1450 feet per mile, and it is most steep in the east-west direction.
3. (a)

   $$\begin{aligned}\nabla \times (\nabla V) &= \nabla \times \left(\frac{\partial V}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\,\hat{\mathbf{z}} \right) \\ &= \hat{\mathbf{x}} \left(\frac{\partial}{\partial y}\frac{\partial V}{\partial z} - \frac{\partial}{\partial z}\frac{\partial V}{\partial y} \right) + \hat{\mathbf{y}} \left(\frac{\partial}{\partial z}\frac{\partial V}{\partial x} - \frac{\partial}{\partial x}\frac{\partial V}{\partial z} \right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial}{\partial y}\frac{\partial V}{\partial x} \right) \\ &= 0\end{aligned}$$

   (b)

   $$\begin{aligned}\nabla \cdot (\nabla \times A) &= \nabla \cdot \left[\hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\ &= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ &= 0\end{aligned}$$

   (c) For convenience: Let A^x denote the component of A in the x direction. Let A^x_x denote $\frac{\partial A^x}{\partial x}$, let A^x_{xy} denote $\frac{\partial}{\partial y}\frac{\partial A^x}{\partial x}$, etc. Then some very tedious brute force calculation gives

   $$\begin{aligned}\nabla \times A &= \hat{\mathbf{x}} \left(A^z_y - A^y_z \right) + \hat{\mathbf{y}} \left(A^x_z - A^z_x \right) + \hat{\mathbf{z}} \left(A^y_x - A^x_y \right) \\ \nabla \times (\nabla \times A) &= \hat{\mathbf{x}} \left(A^y_{xy} - A^x_{yy} - A^x_{zz} + A^z_{xz} \right) + \hat{\mathbf{y}} \left(A^z_{yz} - A^y_{zz} - A^y_{xx} + A^x_{yx} \right) + \hat{\mathbf{z}} \left(A^x_{zx} - A^z_{xx} - A^z_{yy} + A^y_{zy} \right) \\ \nabla \cdot A &= A^x_x + A^y_y + A^z_z \\ \nabla(\nabla \cdot A) &= \hat{\mathbf{x}} \left(A^x_{xx} + A^y_{yx} + A^z_{zx} \right) + \hat{\mathbf{y}} \left(A^x_{xy} + A^y_{yy} + A^z_{zy} \right) + \hat{\mathbf{z}} \left(A^x_{xz} + A^y_{yz} + A^z_{zz} \right) \\ \nabla^2 A &= \hat{\mathbf{x}} \nabla^2 A^x + \hat{\mathbf{y}} \nabla^2 A^y + \hat{\mathbf{y}} \nabla^2 A^y \\ &= \hat{\mathbf{x}} \left(A^x_{xx} + A^x_{yy} + A^x_{zz} \right) + \hat{\mathbf{y}} \left(A^y_{xx} + A^y_{yy} + A^y_{zz} \right) + \hat{\mathbf{z}} \left(A^z_{xx} + A^z_{yy} + A^z_{zz} \right)\end{aligned}$$

   It is evident after some trivial subtraction and elimination of equal quantities that we arrive at the desired equality

   $$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A)$$

Let $f(x, y, z) = xyz$ and $A = f\,\hat{\mathbf{x}} + f\,\hat{\mathbf{y}} + f\,\hat{\mathbf{z}}$. If f is differentiated twice with respect to the same variable, it becomes 0. Hence $\nabla^2 A = 0$, because it only contains such derivatives. However, consider the x-component of $\nabla \times (\nabla \times A)$. The derivatives taken twice with respect to the same variable disappear, and we are left with $f_{xy} + f_{xz} = z + y \neq 0$. Thus $\nabla \times (\nabla \times A) \neq 0$ and A satisfies the desired conditions.

4. (a) Let $\mathbf{v} = \mathbf{c}T$. The divergence theorem gives

$$\int_V (\nabla \cdot \mathbf{v})\,\mathrm{d}\tau = \oint_S \mathbf{v} \cdot \,\mathrm{d}\mathbf{a} \implies \int_V (\nabla \cdot \mathbf{c}T)\,\mathrm{d}\tau = \oint_S \mathbf{c}T \cdot \,\mathrm{d}\mathbf{a}$$

$$\nabla \cdot \mathbf{c}T = T(\nabla \cdot \mathbf{c}) + \mathbf{c} \cdot (\nabla T) = \mathbf{c} \cdot (\nabla T)$$

Substituting that back in gives

$$\mathbf{c} \cdot \int_V (\nabla T)\,\mathrm{d}\tau = \mathbf{c} \cdot \oint_S T\,\mathrm{d}\mathbf{a} \implies \int_V (\nabla T)\,\mathrm{d}\tau = \oint_S T\,\mathrm{d}\mathbf{a}$$

(b) The divergence theorem gives

$$\int_V (\nabla \cdot (\mathbf{v} \times \mathbf{c}))\,\mathrm{d}\tau = \oint_S (\mathbf{v} \times \mathbf{c}) \cdot \,\mathrm{d}\mathbf{a}$$

$$\nabla \cdot (\mathbf{v} \times \mathbf{c}) = \mathbf{c} \cdot (\nabla \times \mathbf{v}) \quad \text{and} \quad (\mathbf{v} \times \mathbf{c}) \cdot \,\mathrm{d}\mathbf{a} = \mathbf{c} \cdot (\,\mathrm{d}\mathbf{a} \times \mathbf{v}) = -\mathbf{c} \cdot (\mathbf{v} \times \,\mathrm{d}\mathbf{a})$$

Substituting those back in gives

$$\int_V (\mathbf{c} \cdot (\nabla \times \mathbf{v}))\,\mathrm{d}\tau = -\oint_S \mathbf{c} \cdot (\mathbf{v} \times \,\mathrm{d}\mathbf{a}) \implies \mathbf{c} \cdot \int_V (\nabla \times \mathbf{v})\,\mathrm{d}\tau = -\mathbf{c} \cdot \oint_S (\mathbf{v} \times \,\mathrm{d}\mathbf{a}) \implies \int_V (\nabla \times \mathbf{v})\,\mathrm{d}\tau = -\oint_S \mathbf{v} \times \,\mathrm{d}\mathbf{a}$$

(c) The divergence theorem gives

$$\int_V (\nabla \cdot (T\nabla U))\,\mathrm{d}\tau = \oint_S (T\nabla U) \cdot \,\mathrm{d}\mathbf{a}$$

By the product rule,

$$\nabla \cdot (T\nabla U) = T\nabla^2 U + (\nabla T) \cdot (\nabla U)$$

Substituting that back in we get

$$\int_V (T\nabla^2 U + (\nabla T) \cdot (\nabla U))\,\mathrm{d}\tau = \oint_S (T\nabla U) \cdot \,\mathrm{d}\mathbf{a}$$

(d)

$$\int_V (T\nabla^2 U + (\nabla T) \cdot (\nabla U))\,\mathrm{d}\tau = \oint_S (T\nabla U) \cdot \,\mathrm{d}\mathbf{a}$$

Switching U and T, we have

$$\int_V (U\nabla^2 T + (\nabla U) \cdot (\nabla T))\,\mathrm{d}\tau = \oint_S (U\nabla T) \cdot \,\mathrm{d}\mathbf{a}$$

Subtracting the latter from the former gives the desired result,

$$\int_V (T\nabla^2 U - U\nabla^2 T)\,\mathrm{d}\tau = \oint_S (T\nabla U - U\nabla T) \cdot \,\mathrm{d}\mathbf{a}$$

(e) Stokes' theorem gives

$$\int_S (\nabla \times (\mathbf{c}T)) \cdot \,\mathrm{d}\mathbf{a} = \oint_P (\mathbf{c}T) \cdot \,\mathrm{d}\mathbf{l}$$

$$\nabla \times (\mathbf{c}T) = T(\nabla \times \mathbf{c}) = -\mathbf{c} \times (\nabla T) \implies -(\mathbf{c} \times (\nabla T)) \cdot \,\mathrm{d}\mathbf{a} = -(\mathbf{c} \cdot (\nabla T \times \,\mathrm{d}\mathbf{a}))$$

Substitute this back to get

$$\int_S -(\mathbf{c} \cdot (\nabla T \times \,\mathrm{d}\mathbf{a})) = \oint_P \mathbf{c}T \cdot \,\mathrm{d}\mathbf{l} \implies -\mathbf{c} \cdot \int_S (\nabla T) \times \,\mathrm{d}\mathbf{a} = \mathbf{c} \cdot \oint_P T\,\mathrm{d}\mathbf{l}$$

$$\implies \int_S (\nabla T) \times \,\mathrm{d}\mathbf{a} = -\oint_P T\,\mathrm{d}\mathbf{l}$$

5. See end of homework assignment for sketches.

(a)

$$\nabla \cdot \mathbf{A} = \nabla \cdot (r^n \hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 r^n = \frac{1}{r^2}(n+2)r^{n+1} = (n+2)r^{n-1}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial r^n}{\partial \phi}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(-\frac{\partial r^n}{\partial \theta}\right)\hat{\boldsymbol{\phi}} = 0$$

(b)

$$\nabla \cdot \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}r^n\sin\theta = \frac{1}{r\sin\theta}r^n\cos\theta = \frac{r^{n-1}}{\tan\theta}$$
$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left(-\frac{\partial r^n}{\partial\theta}\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial r^{n+1}}{\partial r}\,\hat{\boldsymbol{\phi}} = \frac{1}{r}(n+1)r^n\,\hat{\boldsymbol{\phi}} = (n+1)r^n\,\hat{\boldsymbol{\phi}}$$

When $n = -2$, the divergence of the first function becomes a Dirac delta function. At the origin, the curl of the first function is ill-defined because the components become 0/0. When $n = 0$, the divergence of the second function becomes a Dirac delta function. At the origin, the curl of the second function is ill-defined because one of the components become 0/0.

6. (a)

$$\rho(\mathbf{r}) = 2Q\delta^3(\mathbf{x}-\mathbf{a}) - Q\delta(\mathbf{x}+\mathbf{a})$$

(b)

$$\rho(\mathbf{r}) = \frac{1}{4\pi R^2}Q\delta^3(|\mathbf{r}|-R)$$

(c)

$$\rho(\mathbf{r}) = \frac{1}{4\pi R^2}Q\delta^3(|\mathbf{r}-\mathbf{a}|-R)$$

(d)

$$\rho(\mathbf{r}) = (Q/L)\delta(x)\delta(y)(H(z+L/2)-H(z-L/2)) = (Q/L)\delta(x)\delta(y)\left(\int_{-\infty}^{z+L/2}\delta(s)\,\mathrm{d}s - \int_{-\infty}^{z-L/2}\delta(s)\,\mathrm{d}s\right)$$

H is the Heaviside step function, so this makes the charge density defined only at $-L/2 \le z \le L/2$.

7. (a) $2\tan^{-1}(s/h)$

   (b) $\pi/2$

   (c) By definition, latitude is the angle subtended at the center of the earth, so the subtended angle is 22.5 degrees or 0.3927 radians.

8. Stokes' theorem states

$$\int_S (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint_P \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

We have $\mathbf{v} = ay\,\hat{\mathbf{x}} + bx\,\hat{\mathbf{y}} + cz\,\hat{\mathbf{z}}$ so

$$\nabla \times \mathbf{v} = (b-a)\,\hat{\mathbf{z}}$$

Also, the boundary can be parameterized by the equations $x = R\cos\phi$, $y = R\sin\phi$, $z = 0$, and we can write $\mathbf{v} = aR\sin\theta\sin\phi\,\hat{\mathbf{x}} + bR\sin\theta\cos\phi + cR\cos\theta\,\hat{\mathbf{z}} = aR\sin\phi\,\hat{\mathbf{x}} + bR\cos\phi\,\hat{\mathbf{y}}$ because $\theta = \pi/2$, $r = R$ along the boundary. The integral around the boundary is

$$\oint_P \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} R^2(a+b)\sin\phi\cos\phi\,\mathrm{d}\phi = 0$$

   (a) Using the interior disk, $\mathrm{d}\mathbf{a} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}}$, so $(\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = ((b-a)\,\hat{\mathbf{z}}) \cdot \mathrm{d}\mathbf{a} = 0$, so the surface integral is 0. This satisfies Stokes' theorem, as expected.

   (b) We wish to integrate over the northern hemisphere. The surface element is $\mathrm{d}a = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. This will be multiplied with $R(b-a)$ and integrated from $0 \le \phi \le 2\pi$, $0 \le \theta \le \pi/2$. Reminder: incomplete - finish this later!

9.

$$\nabla \cdot \mathbf{v} = yz + xz + xy$$
$$\int_V (\nabla \cdot \mathbf{v})\,\mathrm{d}\tau = \int_0^1\int_0^1\int_0^1 (yz+xz+xy)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1\int_0^1 \frac{1}{2}(2xy+x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 (x+1/4)\,\mathrm{d}x = 3/4$$

Along the six faces of the cube, there are six integrals. Three of them vanish because one of the coordinates is 0 so $xyz = 0$. The other three are equivalent by symmetry, so their sum is

$$3\int_0^1\int_0^1 yz\,\mathrm{d}y\,\mathrm{d}z = 3/4$$

As expected, the two quantities are equal, so the divergence theorem holds.