Problem Set 1: Vector fields, derivatives and integrals

1. Griffiths Problem 1.10 (same in Edition 3 and 4).
2. The height of Capitol Hill (in feet) is roughly given by

$$h(x, y) = 426 - 450y^2 - 180y - 1250x^2 \tag{1}$$

   where x, y are measured with respective to the Volunteer Park Conservatory (in miles).

   a) How far is the top of the hill from the Volunteer Park Conservatory?
   b) How high is the hill?
   c) How steep is the slope (in feet per mile) where it hits Lake Union (elevation 23 ft) due west of the peak? In what direction is the slope steepest at that point?

3. Show that

   i) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} V) = 0$ for any scalar field $V(\boldsymbol{r}, t)$.
   ii) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ for any vector field $\boldsymbol{A}(\boldsymbol{r}, t)$
   iii) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$

   Construct a vector field $\boldsymbol{A}$ such that $\nabla^2 \boldsymbol{A} = 0$ but $\nabla \times (\nabla \times \mathbf{A}) \neq \mathbf{0}$.

4. Griffiths Problem 1.61 (Edition 4. This is 1.60 in Edition 3.)
5. Sketch in the XY and YZ planes and find the divergence and curl of the following vector fields:

   i) $\boldsymbol{A} = r^n \hat{\boldsymbol{r}}$ where $\hat{\boldsymbol{r}}$ is the radial unit vector.
   ii) $\boldsymbol{A} = r^n \hat{\boldsymbol{\theta}}$ where $\hat{\boldsymbol{\theta}}$ is the polar angle unit vector.

   In which case(s) do you get a Dirac $\delta-$function at the origin? In which cases is the result ill-defined at the origin?

6. Using $\delta-$functions, write expressions for the volume charge density ρ of the following charge distributions each having total charge Q:

   i) a point charge $+2Q$ at $+\boldsymbol{a}$ and another $-Q$ at $-\boldsymbol{a}$;
   ii) a uniform thin spherical shell of radius R centered at the origin;
   iii) a uniform thin spherical shell of radius R centered at $\boldsymbol{a}$;
   iv) a uniform line charge along the z-axis between $z = -L/2$ and $L/2$;

7. What is the solid angle subtended at the origin by

   i) the interior of the circle $s = a$, $z = h$ in cylindrical polar coordinates;
   ii) the quarter-plane $z = h$, $x > 0$, $y > 0$ in cartesian coordinates;
   iii) the polar cap of the earth (above the arctic circle $\theta = 22.5\,\text{deg}$ at $R_{Earth} = 6380$ km) (when the origin is at the center of the earth).

8. Check Stokes' theorem for $\boldsymbol{v} = ay\hat{\boldsymbol{x}} + bx\hat{\boldsymbol{y}} + cz\hat{\boldsymbol{z}}$ using the circle $r = R, \theta = \pi/2$ (in spherical coordinates) and two different surfaces: (i) the interior disk $(r < R, \theta = \pi/2)$; and (ii) the upper hemisphere $(r = R, \theta < \pi/2)$.
9. Check the divergence theorem for $\boldsymbol{v} = xyz\hat{\boldsymbol{x}} + xyz\hat{\boldsymbol{y}} + xyz\hat{\boldsymbol{z}}$ on the positive unit cube (ie. the cube whose main diagonal goes from the origin $(0, 0, 0)$ to $(1, 1, 1)$).