Problem Set 2

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1 Problem Set

1. (a)

$$
\begin{aligned}
\mathbf{E} &= ar^n\,\hat{\mathbf{r}} + (bx+cy)\,\hat{\mathbf{x}} + (by+dx)\,\hat{\mathbf{y}} \\
&= ar^n\,\hat{\mathbf{r}} + (br\sin\theta\cos\phi + cr\sin\theta\sin\phi)(\sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}) \\
&\quad + (br\sin\theta\sin\phi + dr\sin\theta\cos\phi)(\sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}) \\
&= (ar^n + br\sin^2\theta(\sin^2\phi+\cos^2\phi) + r(c+d)\sin^2\theta\sin\phi\cos\phi)\,\hat{\mathbf{r}} \\
&\quad + (br\sin\theta\cos\theta(\sin^2\phi+\cos^2\phi) + r(c+d)\sin\theta\cos\theta\sin\phi\cos\phi)\,\hat{\boldsymbol{\theta}} \\
&\quad + r\sin\theta(d\cos^2\phi + c\sin^2\phi)\,\hat{\boldsymbol{\phi}} \\
&= (ar^n + r\sin^2\theta(b+(c+d)\sin\phi\cos\phi))\,\hat{\mathbf{r}} + r\sin\theta\cos\theta(b+(c+d)\sin\phi\cos\phi)\,\hat{\boldsymbol{\theta}} + r\sin\theta(d\cos^2\phi + c\sin^2\phi)\,\hat{\boldsymbol{\phi}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}E_\phi \\
&= \frac{1}{r^2}\left[\left(a(n+2)r^{n+1}\right) + 3r^2\sin^2\theta(b+(c+d)\sin\phi\cos\phi)\right] \\
&\quad + \frac{1}{r\sin\theta}\left[r\left(\frac{1}{4}(3\sin(3\theta)-\sin\theta)\right)(b+(c+d)\sin\phi\cos\phi)\right] \\
&\quad + \frac{1}{r\sin\theta}\left[-r(c+d)\sin\theta\sin(2\phi)\right] \\
&= \left[\left(a(n+2)r^{n-1}\right) + 3\sin^2\theta(b+(c+d)\sin\phi\cos\phi)\right] + \left[\left(\frac{3}{2}\cos(2\theta)+\frac{1}{2}\right)(b+(c+d)\sin\phi\cos\phi)\right] \\
&\quad + \left[-(c+d)\sin(2\phi)\right] \\
&= a(n+2)r^{n-1} + 2b
\end{aligned}
$$

$$\rho(\mathbf{r}) = \epsilon_0(\nabla\cdot\mathbf{E}) = \epsilon_0\left(a(n+2)r^{n-1}+2b\right)$$

When $n=-2$, $\rho = 2b\epsilon_0$, so the charge density is constant throughout all of space.

(b)

$$
\begin{aligned}
Q_{r<R} &= \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^R r^2\,\mathrm{d}r\,\rho(r) = 4\pi\epsilon_0\int_0^R\left(a(n+2)r^{n+1}+2br^2\right)\,\mathrm{d}r \\
&= 4\pi\epsilon_0\int_0^R 2br^2\,\mathrm{d}r + 4\pi\epsilon_0 a\int_0^R(n+2)r^{n+1}\,\mathrm{d}r = \frac{8}{3}R^3b\pi\epsilon_0 + 4R^{n+2}a\pi\epsilon_0
\end{aligned}
$$

(c) The curl of $\mathbf{E}$ is equal to $\nabla\times\mathbf{E} = (d-c)\,\hat{\mathbf{z}}$, so when $c=d$, the curl is equal to 0, and $\mathbf{E}$ is an irrotational field.

2. (a) Suppose that the cylinder is oriented lengthwise along the z-axis and perpendicular to the xy plane so that the cylinder ranges from $z=-L/2$ to $z=+L/2$. Because the cylinder is symmetric around the z axis, the electric field at any point on the z axis can be thought to point only in the z direction. Suppose we have a disk of radius R centered on the z axis with charge-per-unit-area σ. First, we will calculate the electric field due to such a disk; secondly, we will sum those disks to arrive at the electric field of the cylinder.

Suppose that we are a distance z away from the center of the disk. On the disk, we can say that $\mathrm{d}q = \sigma\,\mathrm{d}a = 2\sigma\pi r\,\mathrm{d}r$ where r is measured from the center of the disk outward. Also, $\mathscr{r}^3 = \left(z^2+r^2\right)^{3/2}$. Finally, the z component of $\boldsymbol{\mathscr{r}}$ will be given by $\mathscr{r}\cos\theta = \mathscr{r}\,(z/\mathscr{r}) = z$, where θ is the angle formed by $\boldsymbol{\mathscr{r}}$ and the z-axis. Then, the electric field at $\mathbf{r} = z\,\hat{\mathbf{z}}$ due to such a disk is given by

$$
\begin{aligned}
E_z &= \frac{1}{4\pi\epsilon_0}\int\frac{1}{\mathscr{r}^3}\boldsymbol{\mathscr{r}}\,\mathrm{d}q = \frac{1}{4\pi\epsilon_0}\int_0^R\frac{z2\sigma\pi r\,\mathrm{d}r}{(z^2+r^2)^{3/2}} = \frac{z\sigma}{2\epsilon_0}\int_0^R\frac{r\,\mathrm{d}r}{(z^2+r^2)^{3/2}} = \frac{z\sigma}{2\epsilon_0}\left(-\frac{1}{\sqrt{r^2+z^2}}\right)\Bigg|_0^R \\
&= \frac{z\sigma}{2\epsilon_0}\left(\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right) = \frac{\sigma}{2\epsilon_0}\left(\operatorname{sgn} z - \frac{z}{\sqrt{R^2+z^2}}\right)
\end{aligned}
$$

where sgn denotes the sign or signum function. Now for the cylinder, we perform the integration

$$E_z = \int_{-L/2}^{L/2} \mathrm{d}z' \, \frac{\rho}{2\epsilon_0} \left(\operatorname{sgn}(z - z') - \frac{z - z'}{\sqrt{R^2 + (z - z')^2}} \right)$$

Perform the substitution $u = z - z' \implies \mathrm{d}z' = -\,\mathrm{d}u$ to obtain

$$E_z = \frac{\rho}{2\epsilon_0} \left(-(z - z') \operatorname{sgn}(z - z') + \sqrt{R^2 + (z - z')^2} \right) \Big|_{-L/2}^{L/2}$$

$$\implies \mathbf{E} = \frac{\rho}{2\epsilon_0} \left((z + L/2) \operatorname{sgn}(z + L/2) - (z - L/2) \operatorname{sgn}(z - L/2) - \sqrt{R^2 + (z + L/2)^2} + \sqrt{R^2 + (z - L/2)^2} \right) \hat{\mathbf{z}}$$

As expected, $E_z(0) = 0$ and $\operatorname{sgn} E_z = \operatorname{sgn} z$.

(b) In the infinite limit the cylinder seems like a point charge with charge $\pi R^2 L \rho$, so we expect

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\pi R^2 L \rho}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} = \frac{R^2 L \rho}{4\epsilon_0 r^2} \hat{\mathbf{r}}$$

If we let $u_{+-} = 2R/(2z \pm L)$, then we can rewrite the electric field as

$$E_z = \frac{\rho}{\epsilon_0} \left((2z + L) \operatorname{sgn}(z + L/2) - (2z - L) \operatorname{sgn}(z - L/2) + (2z - L)\sqrt{1 + u_-^2} - (2z + L)\sqrt{1 + u_+^2} \right)$$

First, $(2z + L) \operatorname{sgn}(z + L/2) - (2z - L) \operatorname{sgn}(z - L/2) = 2L$. In the limit where z is large, u is nonzero but very small. As such, we can consider the Taylor expansion of the square roots. We will discard all terms above the 3rd order.

$$(2z - L)\sqrt{1 + u_-^2} \approx (2z - L) \left(1 + \frac{1}{2} \frac{2R}{2z - L} - \frac{1}{8} \frac{4R^2}{(2z - L)^2} + \frac{1}{16} \frac{8R^3}{(2z - L)^2} \right) = 2z - L + R - \frac{1}{2} \frac{R^2}{2z - L}$$

$$(2z + L)\sqrt{1 + u_+^2} \approx (2z + L) \left(1 + \frac{1}{2} \frac{2R}{2z + L} - \frac{1}{8} \frac{4R^2}{(2z + L)^2} + \frac{1}{16} \frac{8R^3}{(2z + L)^2} \right) = 2z + L + R - \frac{1}{2} \frac{R^2}{2z + L}$$

The difference is

$$(2z - L)\sqrt{1 + u_-^2} - (2z + L)\sqrt{1 + u_+^2} \approx -2L + \frac{1}{2} R^2 \left(\frac{2L}{L^2 - 4z^2} \right) \approx -2L + \frac{R^2 L}{4z^2}$$

because L^2 can be considered to be 0. Substituting that back into our original expression, the electric field is

$$\mathbf{E} = \frac{\rho}{\epsilon_0} \frac{R^2 L}{4z^2} \hat{\mathbf{z}} = \frac{R^2 L \rho}{4\epsilon_0 z^2} \hat{\mathbf{z}}$$

for large z. This agrees with our earlier approximation.

3. Consider the spherical surface of radius r centered on the origin. The flux through it is

$$\Phi_{\text{sphere}} = \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \, \mathrm{d}\phi \, \mathrm{d}\theta = \frac{q}{\epsilon_0}$$

Now consider the surface of a cube of side length $2a$ centered on the origin. In particular, consider the flux through the face of the cube with $z = +a$, parallel to the xy-plane.

$$\Phi_{\text{yz}+} = \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \int_{-a}^{a} \int_{-a}^{a} \frac{qa}{4\pi\epsilon_0} \frac{1}{(x^2 + y^2 + a^2)^{3/2}} \, \mathrm{d}x \, \mathrm{d}y = \frac{qa}{4\pi\epsilon_0} \int_{-a}^{a} \left(\frac{x}{(a^2 + y^2)\sqrt{x^2 + y^2 + a^2}} \right) \Bigg|_{-a}^{a} \mathrm{d}y$$

$$= \frac{qa}{4\pi\epsilon_0} \int_{-a}^{a} \frac{2a}{(a^2 + y^2)\sqrt{2a^2 + y^2}} \, \mathrm{d}y = \frac{qa^2}{2\pi\epsilon_0} \frac{1}{a^2} \left[\tan^{-1} \left(\frac{y}{\sqrt{2a^2 + y^2}} \right) \right] \Bigg|_{-a}^{a} = \frac{qa^2}{2\pi\epsilon_0} \frac{1}{a^2} \frac{\pi}{3} = \frac{q}{6\epsilon_0}$$

Due to symmetry considerations, the flux through each face of the cube must be the same, so the flux through the entire cube is simply

$$\Phi_{\text{cube}} = 6\Phi_{\text{yz}+} = \frac{q}{\epsilon_0}$$

Calculations of the flux through both surfaces result in the same answer, and Gauss's law holds. If the charge were slightly displaced, the flux through an enclosing would not change as long as the charge remains entirely within the surface.

4. (a) Consider a spherical Gaussian surface of radius r. Then Gauss's law gives

$$|\mathbf{E}|4\pi r^2 = \frac{1}{\epsilon_0}4\pi\int_0^r r'^2 a(r'-R)^2\,\mathrm{d}r' = \frac{1}{\epsilon_0}4\pi\int_0^r (ar'^4 - 2Rar'^3 + R^2ar^2)\,\mathrm{d}r' = \frac{4\pi}{\epsilon_0}\left(\frac{1}{5}ar^5 - \frac{1}{2}Rar^4 + \frac{1}{3}R^2ar^3\right)$$

Then inside the sphere, where $r < R$,

$$\mathbf{E}_{r<R} = \frac{1}{\epsilon_0}\left(\frac{1}{5}ar^3 - \frac{1}{2}Rar^2 + \frac{1}{3}R^2ar\right)\hat{\mathbf{r}}$$

Outside of the sphere, for $r \geq R$, the charge is spherically symmetric, so we can treat it as a point charge at the origin, where

$$Q_{\text{tot}} = 4\pi\int_0^R r^2a(r-R)^2\,\mathrm{d}r = \frac{2}{15}\pi aR^5$$

$$\mathbf{E}_{r\geq R} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{tot}}}{r^2}\,\hat{\mathbf{r}} = \frac{aR^5}{30\epsilon_0 r^2}\,\hat{\mathbf{r}}$$

(b) Consider a "Gaussian pillbox" with thickness $2z$ (so that it extends z above the xy plane and z below it as well) and lid area α. Then the enclosed charge is $Q_{\text{enc}} = \rho V = 2abz$. Due to symmetry, the electric field necessarily only points in the positive or negative z direction. Thus,

$$\int_S \mathbf{E}\cdot\,\mathrm{d}\mathbf{a} = 2|\mathbf{E}|a = \frac{2abz}{\epsilon_0}$$

$$\mathbf{E}_{-h<z<h} = \frac{bz}{\epsilon_0}\,\hat{\mathbf{z}}$$

When $|z| \geq h$, we see that $Q_{\text{enc}} = bh/\epsilon_0$. Also, $\int \mathrm{d}\mathbf{a}$ remains the same, because the area of the top and bottom lids does not change. As such, the electric field is constant:

$$\mathbf{E}_{z\geq h} = \frac{bh}{\epsilon_0}\,\hat{\mathbf{z}} \quad \text{and} \quad \mathbf{E}_{z\leq -h} = -\frac{bh}{\epsilon_0}\,\hat{\mathbf{z}}$$

(c) The electric field due to an infinite plane parallel to the xy plane is $\mathbf{E} = \sigma/2\epsilon_0\,\hat{\mathbf{z}}$. As such, at any location that is not located within the slab, the electric field from this slab is

$$\mathbf{E}_{|z|>h} = \int_{-h}^{h}\frac{bz}{2\epsilon_0}\,\mathrm{d}z = 0$$

Now suppose we have a Gaussian pillbox with area a, one end located at $z = z'$, $-h \leq z' \leq h$, and the other end somewhere outside of the slab, $z > h$. Then the enclosed charge is

$$Q_{\text{enc}} = \int_{z'}^{h} abz\,\mathrm{d}z = \frac{1}{2}ab\left(h^2 - z'^2\right)$$

By Gauss's law,

$$\int_S \mathbf{E}\cdot\,\mathrm{d}\mathbf{a} = |\mathbf{E}|a = \frac{ab(h^2 - z'^2)}{2\epsilon_0}$$

The field is pointing down within the slab, so

$$\mathbf{E}_{-h\leq z\leq h} = -\frac{ab(h^2 - z^2)}{2\epsilon_0}\,\hat{\mathbf{z}}$$

(d) Suppose we have a Gaussian cylinder centered inside the rod with radius s and length l. Then the enclosed charge is $Q_{\text{enc}} = \pi s^2 l\rho$. The electric field must point radially outward, so Gauss's law gives

$$|\mathbf{E}|2\pi sl = \frac{\pi s^2 l\rho}{\epsilon_0}$$

$$\mathbf{E}_{s\leq R} = \frac{s\rho}{2\epsilon_0}\hat{\mathbf{s}}$$

Outside of the cylinder of charge, $Q_{\text{enc}} = \pi R^2 l\rho$, so we have

$$\mathbf{E}_{s>R} = \frac{R^2\rho}{2s\epsilon_0}\hat{\mathbf{s}}$$

These results should not agree with those of problem 2 in any limit.

5. Since $a \ll R$, we can approximate the drilled-out hole as a cylinder with length $2R$, and the total charge is

$$Q_{\text{tot}} = \rho \left(\frac{4}{3}\pi R^3 - 2R\pi a^2 \right)$$

Very far away from the bead, it just looks like a point charge, so

$$\mathbf{E}_{\text{r}\gg\text{R}} = \frac{1}{4\pi\epsilon_0 r^2}\rho \left(\frac{4}{3}\pi R^3 - 2R\pi a^2 \right) \hat{\mathbf{r}} = \frac{\rho}{\epsilon_0 r^2} \left(\frac{1}{3}R^3 - \frac{1}{2}Ra^2 \right) \hat{\mathbf{r}}$$

We can consider, separately, the complete sphere with no hole and a cylinder with charge density $-\rho$ such that, adding together the charge densities together, we arrive at our "bead" charge density. First, consider the electric field inside a uniformly charged sphere. Gauss's law gives

$$\mathbf{E}_{\text{sphere},r<R} = \frac{\rho r}{3\epsilon_0} \hat{\mathbf{r}}$$

Let us say that the hole is drilled into the sphere along the z-axis. Knowing that $r \ll R$, the cylinder "looks" as though it is infinitely long, and it will not vary greatly along the axis of the cylinder. As such, the electric field is given by

$$\mathbf{E}_{\text{cylinder}} = -\frac{s\rho}{2\epsilon_0} \hat{\mathbf{s}}$$

We can take the sum to arrive at an approximation that is valid strictly inside the cavity, close to the center:

$$\mathbf{E} = \frac{\rho r}{3\epsilon_0} \hat{\mathbf{r}} - \frac{s\rho}{2\epsilon_0} \hat{\mathbf{s}}$$

2 Tutorial

1. (a)

$$\begin{aligned} Q_{\text{tot}} &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \, R^2 \sigma_0 \sin^2\theta \cos^2\phi = R^2\sigma_0 \int_0^{2\pi} \cos^2\phi \, \mathrm{d}\phi \int_0^{\pi} \sin^2\theta \, \mathrm{d}\theta = R^2 \sigma_0 \pi \frac{\pi}{2} \\ &= \frac{1}{2} R^2 \pi^2 \sigma_0 \end{aligned}$$

   (b) The charge distribution is always zero or positive on the surface of the sphere, oscillating between 0 and a maximum of σ_0. For any given value of ϕ, the charge density is zero at the top of the sphere, increases and reaches a maximum at the equator, and then decreases back to zero at the bottom. For any given value of θ, the charge distribution is periodic along any given "slice" of the sphere parallel to the xy plane; it goes through two full periods of oscillation between zero and a maximum value.

   (c) I would set up an integral expression for the electric field and solve it numerically with a computer. Since the charge density is given in terms of θ and ϕ, it is probably easiest to integrate over spherical coordinates.

   (d) It would not change and it would be equally as difficult, because in the previous case I still integrate over spherical coordinates (because that is how the charge density is defined). The only thing that would become simpler is that the calculation of the distance from the point to any position on the sphere becomes slightly–just slightly–easier.

2. The electron will be attracted to the ring. Since the ring is a circle, the position of the electron will vary only along the s and z axes (in cylindrical coordinates). That is to say, its motion will solely be along a certain plane that intersects the center of the ring, because of the symmetry of the ring. If it starts along the ring's central axis, then it will eventually come to rest at the center of the ring, given infinite time (otherwise the electric field is nonzero and it experiences a force). If it starts elsewhere, it will eventually come to rest at the point on the ring closest to its initial starting location, given infinite time, or perhaps on the position on the ring opposite to that position, depending on the specific initial conditions.