ITERATING THE LOGARITHMIC FUNCTION

XIANGLONG NI, JIU DING

Abstract. We analyze the periodic points of the function $f(x) = \log_a |x|$, where $a = e^{1/e}$. Surprisingly, the periodic points of this function are uniquely characterized by the signs of their iterates, so we are able to establish a bijection and then consequently determine that the number of periodic points of period n is equal to 2^n. From there, recursion can then be used to determine how many of these are actually proper periodic points of period n.

1. Introduction

For convenience and brevity, we will denote some commonly used functions as follows:

$$\begin{aligned} f(x) &= \log_a |x| \\ h(x) &= \log_a x \\ g_i(x) &= -x \text{ or } x \\ j(x) &= h^{-1}(x) = a^x \end{aligned}$$

We call points that satisfy $f^n(x) = x$ periodic points of period n of f. If it is also true that there exists no k, $1 \leq k < n$, such that $f^k(x) = x$, then we say that x is a proper periodic point of period n of f. In the case that $n = 1$, so that $f(x) = x$, we call x a fixed point of f.

We consider the special case that $a = e^{1/e}$ because it results in exactly two fixed points for f, whereas for bases larger than $e^{1/e}$, f has only one fixed point, and for bases less than $e^{1/e}$ but greater than 1, f has three fixed points.

Consider a period n periodic point of f:

$$\begin{aligned} f^n(x) &= x \\ (f \circ f \circ \cdots \circ f)(x) &= x \\ (h \circ g_n) \circ (h \circ g_{n-1}) \circ \cdots \circ (h \circ g_1)(x) &= x \\ x &= (g_1^{-1} \circ h^{-1}) \circ (g_2^{-1} \circ h^{-1}) \circ \cdots \circ (g_n^{-1} \circ h^{-1})(x) \\ x &= (g_1 \circ j) \circ (g_2 \circ j) \circ \cdots \circ (g_n \circ j)(x) \end{aligned}$$

We will refer to this equation as the descriptive equation of x, because the signs of the g_i characterize the orbit of x: $g_i(x) = x \iff f^{i-1}(x) > 0$ and $g_i(x) = -x \iff f^{i-1}(x) < 0$. Note that the orbit of any periodic point of f will never include 0, as that would imply that the following iteration would be undefined.

Denote $p(x) = (g_1 \circ j) \circ (g_2 \circ j) \circ \cdots \circ (g_n \circ j)(x)$. From the above, it is clear that every period n point has exactly one corresponding $p(x)$. If we can show that every $p(x)$ gives exactly one solution to $p(x) = x$, then we can establish a bijection between the set of all period n points and the set of all possible $p(x)$ of length n.

Let us note a few things about $p(x)$. First, $p(x) \leq j^n(x)$. That is to say, the tower of exponents is maximized when all the signs are positive. Additionally, e is the only fixed point of $j(x) = a^x$, and for any $x < e$, $j(x) < e$ also. Inductively, we deduce that for $x < e$, $p(x) \leq j^n(x) < e$ as well. Furthermore, $p(x)$ is a composition of functions that are either monotonically increasing (a^x and x) or monotonically decreasing ($-x$), so $p(x)$ itself is either monotonically increasing or monotonically decreasing, depending on how many of the $g_i(x)$ are $-x$. These observations will prove useful later.

2. Solving Descriptive Equations

Consider the behavior of $p(x)$ as x tends towards negative and positive infinity. Suppose $g_i(x) = x$ for all $1 \leq i \leq n$. From before, we know that the only solution to $p(x) = x$ in this case is $x = e$. So from this

point onwards, suppose instead that the g_i are not all equal to x. Let m be the largest number such that $g_m(x) = -x$. Then,

$$\lim_{x\to\infty} g_{m+1} \circ j \circ g_{m+2} \circ \cdots \circ j(x) = \infty$$
$$\lim_{x\to\infty} g_m \circ j \circ g_{m+1} \circ j \circ g_{m+2} \circ \cdots \circ j(x) = -\infty$$

If $m = 1$, then the last statement is equivalent to $\lim_{x\to\infty} p(x) = -\infty$, thus $\lim_{x\to\infty} p(x) - x = -\infty$ as well. If $m \neq 1$,

$$\begin{aligned} \lim_{x\to\infty} j \circ g_m \circ j \circ g_{m+1} \circ \cdots \circ j(x) &= 0 \\ \lim_{x\to\infty} p(x) = \lim_{x\to\infty} g_1 \circ j \circ g_2 \circ \cdots \circ j \circ g_{m-1} &\circ (j \circ g_m \circ j \circ g_{m+1} \circ \cdots \circ j(x)) \\ = g_1 \circ j \circ g_2 \circ \cdots \circ j \circ g_{m-1}(0) \end{aligned}$$

which is some finite value less than e, by our earlier observation. Hence, $\lim_{x\to\infty} p(x) - x = -\infty$ once again. Now we consider the behavior of $p(x)$ as x approaches negative infinity.

$$\begin{aligned} \lim_{x\to-\infty} j(x) &= 0 \\ \lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} g_1 \circ j \circ \cdots \circ g_n &\circ (j(x)) \\ = g_1 \circ j \circ \cdots \circ g_n(0) \end{aligned}$$

which is, again, some finite value less than e. Hence, $\lim_{x\to-\infty} p(x) - x = \infty$.

Because $p(x) - x$ is continuous, it follows from the Intermediate Value Theorem that $p(x) - x$ must have at least one root. Furthermore, from our earlier observation, we know that $p(x)$ is either monotonically increasing or decreasing. Since it is bounded by values less than e on both sides, we deduce that $p(x)$ only assumes values less than e, so any solutions to $p(x) = x$ must also be less than e. If we can show that the slope of $p(x) - x$ is negative for $x < e$, meaning that the function is monotonically decreasing, then there must be exactly one solution.

With this in mind, we differentiate $p(x) - x$ using the chain rule:

$$\frac{d}{dx}(p(x) - x) = (g_1' \circ j \circ g_2 \circ \cdots \circ j \circ g_n \circ j(x))(j' \circ g_2 \circ \cdots \circ j \circ g_n \circ j(x)) \cdots (j'(x)) - 1$$

Recall that $g_i(x)$ is either x or $-x$, so $g_i'(x)$ is either 1 or -1, and that $j(x) = a^x$, so $j'(x) = a^x \ln a = j(x) \ln a$. So, we can rewrite the above as

$$\frac{d}{dx}(p(x) - x) = (\ln a)^n g_1' g_2' \cdots g_n' (j \circ g_2 \circ j \cdots \circ g_n \circ j(x))(j \circ g_3 \circ j \cdots \circ g_n \circ j(x)) \cdots (j(x)) - 1$$

If the product of the $g_i'(x)$ is -1, then the expression is negative, and we are done. Suppose, then, that their product is 1. Because we are only concerned with $x < e$, we may once again make use of our earlier observation to find that $0 < j \circ g_k \circ j \cdots \circ g_n \circ j(x) < e$ for every exponential tower in the above expression. Then,

$$\begin{aligned} \frac{d}{dx}(p(x) - x) &= (\ln a)^n g_1' g_2' \cdots g_n' (j \circ g_2 \circ j \cdots \circ g_n \circ j(x))(j \circ g_3 \circ j \cdots \circ g_n \circ j(x)) \cdots (j(x)) - 1 \\ &< \frac{1}{e^n} e^n - 1 \\ &\leq 0 \end{aligned}$$

as desired.

From this, we conclude that every descriptive equation has exactly one unique solution. Hence, we have established a bijection between periodic points and their corresponding descriptive equations, and to count the former, we need only count the latter.

3. Counting the Periodic Points of f

The number of descriptive equations of length n is straightforward to determine. Every $g_i(x)$ has two possibilities, x or $-x$, and there are n such $g_i(x)$, so there are a total of 2^n descriptive equations of length n, each one of which corresponds with a periodic point of period n of f. Thus, there are 2^n period n periodic points of $f(x) = \log_a |x|$. However, these are not necessarily proper periodic points of period n. Although x may satisfy $f^n(x) = x$, it might be the case that $f^k(x) = x$ for some smaller k, $k|n$. All periodic points of period k are also periodic points of period n if $k|n$, since $f^n(x) = (f^k)^m(x)$ for some m.

Recursion can be used to determine the number of proper periodic points of period n by directly subtracting the number of proper periodic points of periods that divide n:

$$b_n = 2^n - \sum_{\substack{k|n \\ 1 \le k < n}} b_k, \text{ for } n \ge 1.$$

The first few values of b_n, calculated in this method, are as follows.

n	1	2	3	4	5	6	7	8	9	10
b_n	2	2	6	12	30	54	126	240	504	990

Note that $n|b_n$ always, as expected, since every period n orbit contains n of the proper periodic points.

4. Conclusion

The periodic points of the function $f(x) = \log_a |x|$, where $a = e^{1/e}$, are closely related to their orbits. In fact, they can be uniquely determined by the signs of their iterates. Given any sequence of n signs, the equation we construct from them will always have exactly one periodic point as a solution. This enables us to establish a bijection between periodic points of f and binary sequences of signs, which are simple to count. Thus, we deduce that there are 2^n periodic points of period n of f. To determine how many of these are actually proper periodic points of period n, we proceed recursively, subtracting the number of proper periodic points of periods that divide into n. In this manner, we not only able to show that periodic points of f exist for any arbitrary period, but we are also able to calculate the exact number, without actually determining the values of the periodic points themselves.