

MS 221 MOCK EXAM (R06)

PART I

Instructions

- (i) You should attempt as many questions as you can in this part of the examination.
- (ii) Part 1 carries 72% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.
- (iii) You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.

Question 1 - 6 marks

- (a) Use the Fibonacci recurrence relation to show that

$$\frac{F_n}{F_{n+1}F_{n+2}} = \frac{1}{F_{n+1}} - \frac{1}{F_{n+2}} \quad \text{for } n = 0, 1, 2, \dots \quad [3]$$

- (b) Hence use the method of telescoping cancellation to show that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{2}{3 \times 5} + \frac{3}{5 \times 8} + \dots + \frac{F_n}{F_{n+1}F_{n+2}} = 1 - \frac{1}{F_{n+2}} \quad \text{for } n = 0, 1, 2, \dots \quad [3]$$

Question 2 - 6 marks

This question concerns the curve with equation

$$9x^2 + 4y^2 - 36x + 8y + 4 = 0 \quad (\text{equation 1})$$

- (a) Show that this curve can be obtained from the conic with equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

by translation, and state the translation required. [3]

- (b) Hence sketch the curve with equation 1, showing its axes of symmetry and the coordinates of its vertices. [3]

Question 3 - 6 marks

The isometry g is defined to be the reflection q_θ in the line $y = \frac{3x}{4}$ followed by the translation $t_{8,6}$.

- (a) Show that g is a glide-reflection. [1]
- (b) Give the rules for q_θ and $t_{8,6}$; hence determine the rule for g . [3]
- (c) Determine the image of each of the points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ under g . [2]

Question 4 - 5 marks

From a group of 9 people, of whom 5 are men and 4 are women, 5 are to be selected to form a committee.

- (a) In how many different ways may this selection be made? [2]
- (b) In what proportion of this total number of possible selections will women outnumber men on the committee? [3]

Question 5 - 7 marks

- (a) Find the fixed points of the function $f(x) = 2x(1 - x)$ where $x \in \mathbb{R}$. [2]
- (b) Draw a sketch of the graph of f , together with the line $y = x$, showing the fixed points, and hence explain whether each fixed point is attracting or repelling. [2]
- (c) Use the gradient criterion to find an interval of attraction for one of the fixed points. [3]

Question 6 - 6 marks

Express the matrix M below in the form QDQ^{-1} , where D is a diagonal matrix (you should calculate the matrix Q^{-1} explicitly):

$$M = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \quad [6]$$

Question 7 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $k(x) = \ln(\arcsin(x))$ ($0 < x < 1$) [2]

(b) $g(t) = t^3 \cos(3t)$ [2]

Question 8 - 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int x^3 \ln(3x) dx \quad (x > 0). \quad [2]$$

(b) Using the substitution $u = \ln(x)$, or otherwise, find the indefinite integral

$$\int \frac{\sec^2(\ln(x))}{x} dx. \quad [3]$$

Question 9 - 4 marks

(a) Find the volume of revolution obtained when the region under the graph of

$y = \frac{1}{x}$, from $x = 1$ to $x = k$ (where $k > 1$ is a constant), is rotated about the x -axis. [3]

(b) Hence show that this volume tends to a finite limit as $k \rightarrow \infty$. [1]

Question 10 - 5 marks

(a) Use appropriate results from the Handbook to find the first three non-zero terms of the Taylor series about $x = 0$ for the function

$$f(x) = \sin(x)\cos(x). \quad [2]$$

(b) By differentiating term by term your answer to (a), show that the Taylor polynomial of degree 4 for $f'(x)$ is the same as that for $\cos(2x)$, and explain why you would expect this to be the case. [3]

Question 11 - 5 marks

Let $z = 8 \exp\left(\frac{i\pi}{3}\right)$.

(a) Express z and \bar{z} in Cartesian form, where \bar{z} is the complex conjugate of z . [3]

(b) Hence, find in Cartesian form $z + \bar{z}$ and $z\bar{z}$. [2]

Question 12 - 4 marks

- (a) Show that 658 324 719 is divisible by 9 but not divisible by 18. [2]
- (b) Find a number x in \mathbb{Z}_{15} such that $x \times_{15} 4 = 5$. [1]
- (c) Give an example of a number x in \mathbb{Z}_{15} which has no multiplicative inverse. [1]

Question 13 - 4 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	p	q	r	s
p	r	s	p	q
q	s	r	q	p
r	p	q	r	s
s	q	p	s	r

- (a) Which element is the identity element of $(G, *)$? [1]
- (b) Write down all the self-inverse elements of $(G, *)$. [1]
- (c) Is $(G, *)$ Abelian? Briefly explain your answer. [1]
- (d) To which of the groups listed in the Handbook is $(G, *)$ isomorphic? [1]

Question 14 - 5 marks

Here are two statements about integers n and m , only one of which is true.

- (A) If n and m are both even then $n^2 - m^2$ is even.
- (B) If $n^2 - m^2$ is even then n and m are both even.

- (a) Which of these statements is false? [1]
- (b) Prove that the statement you have identified in (a) is false. [3]
- (c) What is the name of the style of proof you have used in (b)? [1]

PART II

Instructions

- (i) You should attempt not more than **TWO** questions from this part of the examination.
- (ii) Each question in this part carries 14% of the marks.
- (iii) You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.
- (iv) Show all your workings.

Question 15

- (a) Find a closed form for the sequence given by the following recurrence system:

$$u_0 = 1, u_1 = 2, \quad u_{n+2} = -4u_{n+1} - 4u_n \quad (n = 0, 1, 2, \dots) \quad [6]$$

- (b) Show that this sequence satisfies the identity

$$u_{n+1}u_{n-1} - u_n^2 = -4^{n+1} \quad (n = 1, 2, 3, \dots) \quad [4]$$

- (c) Show that the sequence also satisfies the identity

$$\frac{u_{n+1}}{u_n} = \frac{2 + 4n}{1 - 2n} \quad (n = 0, 1, 2, \dots)$$

and deduce that $\frac{u_{n+1}}{u_n} \rightarrow -2$ as $n \rightarrow \infty$. [4]

Question 16

- (a) Find, in the form $\mathbf{x} \mapsto \mathbf{Ax} + \mathbf{a}$ where \mathbf{A} is a 2x2 matrix and \mathbf{a} is a vector with two components, the rule of the affine transformation $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that maps the points (0, 0), (1, 0) and (0, 1) to the points (3, 2), (5, 1) and (6, 0) respectively. [3]

- (b) Hence find the area of the triangle with vertices (3, 2), (5, 1) and (6, 0). [1]

- (c) Show that the matrix \mathbf{A} that you found in part (a) is self-inverse (that is, that $\mathbf{A} = \mathbf{A}^{-1}$). [2]

- (d) Let $\mathbf{x}_1 = \mathbf{Ax}_0 + \mathbf{a}$, where \mathbf{A} and \mathbf{a} are the matrix and vector from part (a) and \mathbf{x}_0 is an arbitrary two-component vector. Use the result of part (b) to show that $\mathbf{x}_0 = \mathbf{A}(\mathbf{x}_1 - \mathbf{a})$. [3]

- (e) Use the result of part (d) to find, in the form $\mathbf{x} \mapsto \mathbf{Bx} + \mathbf{b}$ where \mathbf{B} is a 2x2 matrix and \mathbf{b} is a two-component vector, the rule of the affine transformation f^{-1} . [2]

- (f) Hence find the images under f of the lines $y = x$ and $y = -x$. [3]

Question 17

Use the graph-sketching strategy of Chapter C1 to sketch the graph of the real function $f(x) = \frac{2x+1}{2(3x+1)(x-1)}$. [14]

Question 18

(a) Prove, using mathematical induction, that

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$$

for all positive integers n .

[You may assume, without proof, the formula $\sum_{i=1}^n i = \frac{1}{2}n(n+1)$.] [8]

(b) Use Fermat's Little Theorem to find the remainder when 221^{2004} is divided by 19.

[6]

[END OF QUESTION PAPER]