MATH2038: Partial Differential Equations
Coursework 2

This piece of coursework counts for 5% of your final mark. Work is to be handed in to the Faculty Office (Murray Building, 58) by **15.00 on Friday, 2 May 2014**. *You must collect a cover sheet from the Ketley Room (on level 4 of the Mathematics Building) or from the Foyer of the Faculty office, and fill in all the relevant details* **before** *handing your work in and obtaining a receipt. The Faculty Office is open 09.00-17.00 for early hand-in.*

Marks for late work will be reduced by 10% for each working day, or part of a working day, after the deadline. No marks will be obtained for submissions that are later than 5 working days. If you have difficulties in handing in the work by this deadline you should contact me (D.Nucinkis@soton.ac.uk) **well before** *the deadline.*

You may discuss the problems with others, but you must write up the solutions independently.

We will construct and solve a model for the soil temperature underground, looking at how it varies with the seasons. This introduces an ansatz for finding periodic solutions.

We will assume that the temperature in the ground is a function of time t and depth x only. We will further assume that a reasonable approximation to the temperature at the surface (ground level, given by $x = 0$) is

$$u(0,t) = T_0 + A_0 \cos(\omega t) . \tag{1}$$

Here T_0 is the average temperature, A_0 is the amplitude of the seasonal temperature variation, and ω the frequency, chosen to make the period exactly one year. Explicitly (we will use centimetre-gram-second units throughout) this gives

$$\omega = \frac{2\pi}{\text{year}} = \frac{2\pi}{31\,557\,341}\,\text{s}^{-1} \approx 1.991 \times 10^{-7}\,\text{s}^{-1}. \tag{2}$$

We will assume that the temperature satisfies the heat equation

$$\frac{\partial U}{\partial t} = \kappa^2 \frac{\partial^2 U}{\partial x^2}, \tag{3}$$

where for convenience we introduce the temperature deviation $U(x,t) = u(x,t) - T_0$. The one boundary condition we have is

$$U(0,t) = A_0 \cos(\omega t) . \tag{4}$$

We will also assume that the thermal diffusivity of the soil takes the (uniform) value

$$\kappa^2 \approx 5 \times 10^{-3}\text{cm s}^{-1}. \tag{5}$$

1. **[3 marks]** We are only interested in periodic solutions,

$$U(x,t) = V(x)\cos(\omega t) + W(x)\sin(\omega t)\,. \tag{6}$$

Show that we may regard U to be the real part of the complex function $\tilde{U}$,

$$\tilde{U}(x,t) = X(x)e^{\mathrm{i}\omega t}, \tag{7}$$

where X may be complex, which obeys the heat equation and boundary condition given by

$$\frac{\partial \tilde{U}}{\partial t} = \kappa^2 \frac{\partial^2 \tilde{U}}{\partial x^2}, \qquad \tilde{U}(0,t) = A_0 e^{\mathrm{i}\omega t}. \tag{8}$$

2. **[3 marks]** Show that the general solution for X is

$$X(x) = c_1 \exp\left[-\alpha\sqrt{\frac{\omega}{2\kappa^2}}x\right] + c_2 \exp\left[\alpha\sqrt{\frac{\omega}{2\kappa^2}}x\right], \tag{9}$$

and find the value of α.

3. **[5 marks]** We are interested in solutions that are bounded for all depths, and in particular as $x \to \infty$. Use this to show that the solution for the temperature at a time t and depth x is

$$U(x,t) = A_0 \exp\left[-\sqrt{\frac{\omega}{2\kappa^2}}x\right] \cos\left[\omega t - \sqrt{\frac{\omega}{2\kappa^2}}x\right]. \tag{10}$$

4. **[6 marks]** The solution given by equation (10) clearly has an amplitude $A(x)$ and a phase "lag" $\gamma(x)$ that depend on the depth x. Find the depths (both the exact expression and the approximate numerical value in centimetres) where

   (a) the seasonal variation given by the amplitude A is half that of the surface, A_0;

   (b) the seasons "reverse" and it is hottest in the middle of winter, i.e. the phase lag $\gamma = \pi$.

5. **[3 marks]** Produce a 3-D plot using Maple, Mathematica or some other software that clearly shows the results of Q4b: where the surface temperature has its peaks, the temperature at the depth you found should have troughs and vice-versa. Where necessary for clarity, describe any aspects of your plot.