FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 1

SPECIMEN

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

• Do not open this examination paper until instructed to do so.
• Answer all questions.
• Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
• A graphic display calculator is required for this paper.
• A clean copy of the Mathematics HL and Further Mathematics HL formula booklet is required for this paper.
• The maximum mark for this examination paper is [150 marks].
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

Using l’Hôpital’s Rule, determine the value of

\[ \lim_{x \to 0} \frac{\tan x - x}{1 - \cos x}. \]

2. [Maximum mark: 8]

(a) Show that the following vectors form a basis for the vector space \( \mathbb{R}^3 \).

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} \quad \begin{pmatrix}
2 \\
3 \\
1
\end{pmatrix} \quad \begin{pmatrix}
5 \\
2 \\
5
\end{pmatrix} \]

[3 marks]

(b) Express the following vector as a linear combination of the above vectors.

\[
\begin{pmatrix}
12 \\
14 \\
16
\end{pmatrix}
\]

[5 marks]

3. [Maximum mark: 10]

The positive integer \( N \) is represented by 4064 in base \( b \) and 2612 in base \( b + 1 \).

(a) Determine the value of \( b \). [4 marks]

(b) Find the representation of \( N \)

(i) in base 10;

(ii) in base 12. [6 marks]
4. [Maximum mark: 11]

The weights of potatoes in a shop are normally distributed with mean 98 grams and standard deviation 16 grams.

(a) The shopkeeper places 100 randomly chosen potatoes on a weighing machine. Find the probability that their total weight exceeds 10 kilograms. [3 marks]

(b) Find the minimum number of randomly selected potatoes which are needed to ensure that their total weight exceeds 10 kilograms with probability greater than 0.95. [8 marks]

5. [Maximum mark: 11]

(a) The point $T(at^2, 2at)$ lies on the parabola $y^2 = 4ax$. Show that the tangent to the parabola at $T$ has equation $y = \frac{x}{a} + at$. [3 marks]

(b) The distinct points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p, q \neq 0$, also lie on the parabola $y^2 = 4ax$. Given that the line (PQ) passes through the focus, show that

(i) $pq = -1$;

(ii) the tangents to the parabola at P and Q, intersect on the directrix. [8 marks]

6. [Maximum mark: 7]

The group $\{G, +\}$ is defined by the operation of addition on the set $G = \{2n \mid n \in \mathbb{Z}\}$. The group $\{H, +\}$ is defined by the operation of addition on the set $H = \{4n \mid n \in \mathbb{Z}\}$. Prove that $\{G, +\}$ and $\{H, +\}$ are isomorphic.

7. [Maximum mark: 9]

(a) Given that $a \equiv b \mod p$, show that $a^n \equiv b^n \mod p$ for all $n \in \mathbb{Z}^+$. [4 marks]

(b) Show that $29^{13} + 13^{29}$ is exactly divisible by 7. [5 marks]

Turn over
8. [Maximum mark: 8]

Consider the infinite series \( S = \sum_{n=1}^{\infty} (-1)^{n+1} \sin \left( \frac{1}{n} \right) \).

(a) Show that the series is conditionally convergent but not absolutely convergent. [6 marks]

(b) Show that \( S > 0.4 \). [2 marks]

9. [Maximum mark: 8]

Consider the system of equations

\[
\begin{pmatrix}
1 & -1 & 2 \\
2 & 2 & -1 \\
3 & 5 & -4 \\
3 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
k \\
\end{pmatrix}
=
\begin{pmatrix}
5 \\
3 \\
1 \\
k \\
\end{pmatrix}.
\]

(a) By reducing the augmented matrix to row echelon form,

(i) find the rank of the coefficient matrix;

(ii) find the value of \( k \) for which the system has a solution. [5 marks]

(b) For this value of \( k \), determine the solution. [3 marks]

10. [Maximum mark: 11]

Bill is investigating whether or not there is a positive association between the heights and weights of boys of a certain age. He defines the hypotheses

\[ H_0 : \rho = 0; \quad H_1 : \rho > 0, \]

where \( \rho \) denotes the population correlation coefficient between heights and weights of boys of this age. He measures the height, \( h \) cm, and weight, \( w \) kg, of each of a random sample of 20 boys of this age and he calculates the following statistics.

\[ \sum w = 340, \quad \sum h = 2002, \quad \sum w^2 = 5830, \quad \sum h^2 = 201124, \quad \sum hw = 34150 \]

(This question continues on the following page)
(Question 10 continued)

(a)  (i) Calculate the correlation coefficient for this sample.

(ii) Calculate the \( p \)-value of your result and interpret it at the 1 \% level of significance.  \[8 \text{ marks}\]

(b)  (i) Calculate the equation of the least squares regression line of \( w \) on \( h \).

(ii) The height of a randomly selected boy of this age of 90 cm. Estimate his weight. \[3 \text{ marks}\]

11. \[Maximum \text{ mark: 11}\]

The function \( f \) is defined by \( f(x) = e^x \cos x \).

(a) Show that \( f''(x) = -2e^x \sin x \). \[2 \text{ marks}\]

(b) Determine the Maclaurin series for \( f(x) \) up to and including the term in \( x^4 \). \[5 \text{ marks}\]

(c) By differentiating your series, determine the Maclaurin series for \( e^x \sin x \) up to the term in \( x^3 \). \[4 \text{ marks}\]

12. \[Maximum \text{ mark: 14}\]

The matrix \( A \) is given by

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ -4 & -11 & -2 \end{pmatrix}.
\]

(a)  (i) Find the matrices \( A^2 \) and \( A^3 \), and verify that \( A^3 = 2A^2 - A \).

(ii) Deduce that \( A^4 = 3A^2 - 2A \). \[6 \text{ marks}\]

(b)  (i) Suggest a similar expression for \( A^n \) in terms of \( A \) and \( A^2 \), valid for \( n \geq 3 \).

(ii) Use mathematical induction to prove the validity of your suggestion. \[8 \text{ marks}\]

Turn over
13. [Maximum mark: 9]

A sequence \( \{u_n\} \) satisfies the recurrence relation \( u_{n+2} = 2u_{n+1} - 5u_n, \ n \in \mathbb{N} \). Obtain an expression for \( u_n \) in terms of \( n \) given that \( u_0 = 0 \) and \( u_1 = 1 \).

14. [Maximum mark: 13]

The set \( S \) contains the eight matrices of the form

\[
\begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & c
\end{pmatrix}
\]

where \( a, b, c \) can each take one of the values +1 or −1.

(a) Show that any matrix of this form is its own inverse. \([3 \text{ marks}]\)

(b) Show that \( S \) forms an Abelian group under matrix multiplication. \([9 \text{ marks}]\)

(c) Giving a reason, state whether or not this group is cyclic. \([1 \text{ mark}]\)

15. [Maximum mark: 14]

(a) Prove the internal angle bisector theorem, namely that the internal bisector of an angle of a triangle divides the side opposite the angle into segments proportional to the sides adjacent to the angle. \([6 \text{ marks}]\)

(b) The bisector of the exterior angle \( \hat{A} \) of the triangle ABC meets (BC) at P. The bisector of the interior angle \( \hat{B} \) meets [AC] at Q. Given that (PQ) meets [AB] at R, use Menelaus’ theorem to prove that (CR) bisects the angle \( \hat{A}\hat{C}\hat{B} \). \([8 \text{ marks}]\)
FURTHER MATHEMATICS
HIGHER LEVEL
PAPER 2

SPECIMEN

2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

• Do not open this examination paper until instructed to do so.
• Answer all questions.
• Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
• A graphic display calculator is required for this paper.
• A clean copy of the Mathematics HL and Further Mathematics HL formula booklet is required for this paper.
• The maximum mark for this examination paper is [150 marks].
Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 9]

The relation $R$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $(x_1, y_1)R(x_2, y_2)$ if and only if $\frac{x_1}{y_2} = \frac{x_2}{y_1}$.

(a) Show that $R$ is an equivalence relation. \[6 \text{ marks}\]

(b) Determine the equivalence class containing $(x_1, y_1)$ and interpret it geometrically. \[3 \text{ marks}\]

2. [Maximum mark: 10]

The weights of apples, in grams, produced on a farm may be assumed to be normally distributed with mean $\mu$ and variance $\sigma^2$.

(a) The farm manager selects a random sample of 10 apples and weighs them with the following results, given in grams.

82, 98, 102, 96, 111, 95, 90, 89, 99, 101

(i) Determine unbiased estimates for $\mu$ and $\sigma^2$. \[5 \text{ marks}\]

(ii) Determine a 95% confidence interval for $\mu$. \[5 \text{ marks}\]

(b) The farm manager claims that the mean weight of apples is 100 grams but the buyer from the local supermarket claims that the mean is less than this. To test these claims, they select a random sample of 100 apples and weigh them. Their results are summarized as follows, where $x$ is the weight of an apple in grams.

$\sum x = 9831; \sum x^2 = 972578$

(i) State suitable hypotheses for testing these claims.

(ii) Determine the $p$-value for this test.

(iii) At the 1% significance level, state which claim you accept and justify your answer. \[5 \text{ marks}\]
3. [Maximum mark: 15]

In the acute angled triangle ABC, the points E, F lie on [AC], [AB] respectively such that [BE] is perpendicular to [AC] and [CF] is perpendicular to [AB]. The lines (BE) and (CF) meet at H. The line (BE) meets the circumcircle of the triangle ABC at P. This is shown in the following diagram.

(a) (i) Show that CEFB is a cyclic quadrilateral.

(ii) Show that $\overline{HE} = \overline{EP}$. \[7 \text{ marks}\]

The line (AH) meets [BC] at D.

(b) (i) By considering cyclic quadrilaterals show that $\overline{CAD} = \overline{EFH} = \overline{EBC}$.

(ii) Hence show that $[AD]$ is perpendicular to $[BC]$. \[8 \text{ marks}\]
4. **[Maximum mark: 20]**

The binary operation multiplication modulo 9, denoted by $\times_9$, is defined on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(a) Copy and complete the following Cayley table.

<table>
<thead>
<tr>
<th>$\times_9$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>8</td>
<td>3</td>
<td>7</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Show that $\{S, \times_9\}$ is not a group.  

(c) Prove that a group $\{G, \times_9\}$ can be formed by removing two elements from the set $S$.  

(d) (i) Find the order of all the elements of $G$.  

(ii) Write down all the proper subgroups of $\{G, \times_9\}$.  

(iii) Determine the coset containing the element 5 for each of the subgroups in part (d)(ii).  

(e) Solve the equation $4\times_9 x \times_9 x = 1$.  

---

[3 marks]

[1 mark]

[5 marks]

[8 marks]

[3 marks]
5. [Maximum mark: 24]

Consider the differential equation

\[ \frac{dy}{dx} + y \sec x = x (\sec x - \tan x), \text{ where } y = 3 \text{ when } x = 0. \]

(a) Use Euler’s method with a step length of 0.1 to find an approximate value for \( y \) when \( x = 0.3 \). [5 marks]

(b) (i) By differentiating the above differential equation, obtain an expression involving \( \frac{d^2y}{dx^2} \).

(ii) Hence determine the Maclaurin series for \( y \) up to the term in \( x^2 \).

(iii) Use the result in part (b)(ii) to obtain an approximate value for \( y \) when \( x = 0.3 \). [8 marks]

(c) (i) Show that \( \sec x + \tan x \) is an integrating factor for solving this differential equation.

(ii) Solve the differential equation, giving your answer in the form \( y = f(x) \).

(iii) Hence determine which of the two approximate values for \( y \) when \( x = 0.3 \), obtained in parts (a) and (b), is closer to the true value. [11 marks]
6. [Maximum mark: 25]

(a) A connected planar graph has \( e \) edges, \( f \) faces and \( v \) vertices. Prove Euler’s relation, that is \( v + f = e + 2 \). [8 marks]

(b) (i) A simple connected planar graph with \( v \) vertices, where \( v \geq 3 \), has no circuit of length 3. Deduce that \( e \geq 2f \) and therefore that \( e \leq 2v - 4 \).

(ii) Hence show that \( \kappa_{3,3} \) is non-planar. [9 marks]

(c) The graph \( P \) has the following adjacency table, defined for vertices A to H, where each element represents the number of edges between the respective vertices.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(i) Show that \( P \) is bipartite.

(ii) Show that the complement of \( P \) is connected but not planar. [8 marks]
7. [Maximum mark: 22]

The random variable $X$ has cumulative distribution function

$$F(x) = \begin{cases} 
0 & x < 0 \\
\left(\frac{x}{a}\right)^3 & 0 \leq x \leq a \\
1 & x > a 
\end{cases}$$

where $a$ is an unknown parameter. You are given that the mean and variance of $X$ are $\frac{3a}{4}$ and $\frac{3a^2}{80}$ respectively. To estimate the value of $a$, a random sample of $n$ independent observations, $X_1, X_2, \ldots, X_n$, is taken from the distribution of $X$.

(a) (i) **Find an expression for** $c$ **in terms of** $n$ **such that** $U = c \sum_{i=1}^{n} X_i$ **is an unbiased estimator for** $a$.

(ii) **Determine an expression for** $\text{Var}(U)$ **in this case.**

(b) Let $Y$ denote the largest value of $X$ in the random sample.

(i) **Show that** $P(Y \leq y) = \left(\frac{y}{a}\right)^n$, $0 \leq y \leq a$ **and deduce an expression for the probability density function of** $Y$.

(ii) **Find** $E(Y)$.

(iii) **Show that** $\text{Var}(Y) = \frac{3na^2}{(3n+2)(3n+1)^2}$.

(iv) **Find an expression for** $d$ **in terms of** $n$ **such that** $V = dY$ **is an unbiased estimator for** $a$.

(v) **Determine an expression for** $\text{Var}(V)$ **in this case.**

(c) **Show that** $\frac{\text{Var}(U)}{\text{Var}(V)} = \frac{3n+2}{5}$ **and hence state, with a reason, which of** $U$ **or** $V$ **is the more efficient estimator for** $a$.
8. [Maximum mark: 25]

(a) Given that the elements of a $2 \times 2$ symmetric matrix are real, show that

(i) the eigenvalues are real;

(ii) the eigenvectors are orthogonal if the eigenvalues are distinct.  [11 marks]

(b) The matrix $A$ is given by

$$A = \begin{pmatrix} 11 & \sqrt{3} \\ \sqrt{3} & 9 \end{pmatrix}$$

Find the eigenvalues and eigenvectors of $A$.  [7 marks]

(c) The ellipse $E$ has equation $X^TAX = 24$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A$ is as defined in part (b).

(i) Show that $E$ can be rotated about the origin onto the ellipse $E'$ having equation $2x^2 + 3y^2 = 6$.

(ii) Find the acute angle through which $E$ has to be rotated to coincide with $E'$.  [7 marks]